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Embeddings in Spaces of Lipschitz Type, Entropy and Approximation Numbers, and Applications

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We consider the embeddings of certain Besov and Triebel–Lizorkin spaces in spaces of Lipschitz type. The prototype of such embeddings arises from the result of H. Brézis and S. Wainger (1980, *Comm. Partial Differential Equations* **5**, 773–789) about the "almost" Lipschitz continuity of elements of the Sobolev spaces $H_p^{1+n/p}(\mathbb{R}^n)$ when $1<p<\infty$. Two-sided estimates are obtained for the entropy and approximation numbers of a variety of related embeddings. The results are applied to give bounds for the eigenvalues of certain pseudo-differential operators and to provide information about the mapping properties of these operators. © 2000 Academic Press

INTRODUCTION

In a recent paper [13] we studied spaces of Lipschitz type. Such a space, denoted by $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha\geqslant 0$, is defined to be the space of all functions $f\in C(\mathbb{R}^n)$ such that

$$\|f\mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| := \|f\mid L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,\,y\in\mathbb{R}^n\\ 0<|x-y|<1/2}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y||^\alpha} \tag{0.1}$$

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is finite. Corresponding spaces $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ of functions defined on a bounded domain Ω in $\mathbb{R}^n$ were also introduced. The motivation for this was the well-known result of Brézis and Wainger [3] that every function u in the (fractional) Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, where $1<p<\infty$, is "almost" Lipschitz-continuous in the sense that for all x, $y \in \mathbb{R}^n$ with $0<|x-y|<1/2$,

$$|u(x)-u(y)| \leqslant c\,|x-y|\,|\log|x-y||^{1/p'}\,\|u \mid H_p^{1+n/p}(\mathbb{R}^n)\|. \tag{0.2}$$

Here c is a constant independent of x, y and u; as usual, $1/p+1/p'=1$. Inequality (0.2) implies that $H_p^{1+n/p}(\mathbb{R}^n)$ is continuously embedded in $\mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n)$; we write this as

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n). \tag{0.3}$$

In [13] it was shown that this embedding is sharp, by which we mean that if $\alpha<1/p'$, then

$$H_p^{1+n/p}(\mathbb{R}) \not\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n). \tag{0.4}$$

More general assertions of this nature, concerning Triebel–Lizorkin spaces $F^s_{p,q}$ and Besov spaces $B^s_{p,q}$, were also established. Thus if $1<p<\infty$ and $0<q\leqslant\infty$, it was proved that the embedding

$$F^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n) \tag{0.5}$$

is sharp with respect to the exponent $1/p'$; and that if $0<p\leqslant\infty$ and $1<q\leqslant\infty$, then the embedding

$$B^{1+n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/q')}(\mathbb{R}^n) \tag{0.6}$$

is sharp with respect to the exponent $1/q'$. In addition, when Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, two-sided estimates were obtained of the entropy numbers of the embedding

$$id\colon B^{1+n/p}_{p,q}(\Omega) \to \mathrm{Lip}^{(1,-\alpha)}(\Omega), \tag{0.7}$$

where $0<p\leqslant\infty$, $0<q\leqslant\infty$ and $\alpha>\max(1-1/q, 0)$.

The present paper extends [13] in various ways. We improve the above sharpness assertions by allowing $0<p<\infty$ in (0.5) and $0<q\leqslant\infty$ in (0.6). Better upper estimates are obtained of the entropy numbers of *id* in (0.7), and two-sided estimates of the approximation numbers of *id* are derived.

The situation in which these estimates are obtained is *limiting* in the sense that the "differential dimension" of the domain and target spaces is the same. We complement these results by giving two-sided bounds for the entropy and approximation numbers of similar embeddings in non-limiting situations. We also provide two-sided estimates of these numbers for the embeddings of $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ in $\mathrm{Lip}^{(1,-\beta)}(\Omega)$ when $\beta>\alpha>0$, and indeed for a variety of related embeddings, some involving spaces of Zygmund type. As in [13], much of this work rests upon accurate estimates of entropy numbers of embeddings between certain sequence spaces; we also need similar estimates for approximation numbers. Finally we apply the results to certain (pseudo-) differential operators and provide assertions relating to mapping properties of the operators.

1. PRELIMINARIES

Let $\mathbb{R}^n$ be Euclidean n-space and $\langle x\rangle=(2+|x|^2)^{1/2}$, $x\in\mathbb{R}^n$. In a slight abuse of notation we also use $\langle k\rangle$ to stand for $(2+k^2)^{1/2}$ when $k\in\mathbb{N}$.

Given two (quasi-) Banach spaces X and Y, we write $X\hookrightarrow Y$ if $X\subset Y$ and the natural embedding of X in Y is continuous. For non-negative functions f, g: $\mathbb{N}\to\mathbb{R}$, the symbol $f(k)\sim g(k)$ will mean that there are positive numbers c_1, c_2 such that for all $k\in\mathbb{N}$,

$$c_1 f(k)\leqslant g(k)\leqslant c_2 f(k).$$

All unimportant positive constants will be denoted by c, occasionally with subscripts. For any $\varkappa\in\mathbb{R}$ let

$$\varkappa_+=\max(\varkappa,0) \qquad \text{and} \qquad [\varkappa]=\max\{k\in\mathbb{Z}:k\leqslant\varkappa\}. \tag{1.1}$$

Moreover, for $0<r\leqslant\infty$ the number r' is given by $\frac{1}{r'}:=(1-\frac{1}{r})_+$.

Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. If $m\in\mathbb{N}$, we define $C^m(\mathbb{R}^n)=\{f\colon D^\alpha f\in C(\mathbb{R}^n) \text{ for all } |\alpha|\leqslant m\}$. Here D^α are classical derivatives and $C^m(\mathbb{R}^n)$ is endowed with the norm $\|f\mid C^m(\mathbb{R}^n)\|=\sum_{|\alpha|\leqslant m}\|D^\alpha f\mid L_\infty(\mathbb{R}^n)\|$. Recall the concept of the difference operator Δ_h^m, $m\in\mathbb{N}_0$, $h\in\mathbb{R}^n$: Let f be an arbitrary function on $\mathbb{R}^n$; then

$$(\Delta_h^1 f)(x)=f(x+h)-f(x), \qquad (\Delta_h^{m+1}f)(x)=\Delta_h^1(\Delta_h^m f)(x), \tag{1.2}$$

where x, $h\in\mathbb{R}^n$. For convenience we may write Δ_h instead of Δ_h^1.

DEFINITION 1.1. Let $\alpha \geqslant 0$. Then the space $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\|f \mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f \mid L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x, h \in \mathbb{R}^n \\ 0<|h|<1/2}} \frac{|(\Delta_h f)(x)|}{|h| \, |\log |h||^\alpha} \tag{1.3}$$

is finite.

We gave this definition in [13, Definition 1.1]; it was suggested first by Triebel in some unpublished notes. Note that $\mathrm{Lip}^{(1,0)}(\mathbb{R}^n)$ is just the usual space of Lipschitz-continuous functions on $\mathbb{R}^n$.

We introduce the Zygmund spaces $\mathscr{C}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \geqslant 0$, as some counterparts of the spaces $\mathrm{Lip}^{(1,-\alpha)}$.

DEFINITION 1.2. Let $\alpha \geqslant 0$. Then the space $\mathscr{C}^{(1,-\alpha)}(\mathbb{R}^n)$ is defined as the set of all $f \in C(\mathbb{R}^n)$ such that

$$\|f \mid \mathscr{C}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f \mid L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x, h \in \mathbb{R}^n \\ 0<|h|<1/2}} \frac{|(\Delta_h^2 f)(x)|}{|h| \, |\log |h||^\alpha} < \infty.$$

We recall briefly the basic ingredients needed to introduce spaces of type $B^s_{p,q}$ and $F^s_{p,q}$. Leopold studied in [22] spaces of type $B^{(s,b)}_{p,q}$, $b \in \mathbb{R}$, which extend the scale of usual B-spaces in terms of smoothness. To compare related results later we give the more general definition of $B^{(s,b)}_{p,q}$ instead of $B^s_{p,q}$. The Schwartz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Furthermore, $L_p(\mathbb{R}^n)$ with $0<p\leqslant\infty$, is the usual quasi-Banach space with respect to Lebesgue measure. Let $\varphi \in S(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \varphi \subset \{ y \in \mathbb{R}^n : |y| < 2\} \qquad \text{and} \qquad \varphi(x) = 1 \qquad \text{if} \quad |x| \leqslant 1,$$

put $\varphi_0 = \varphi$ and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then since $1 = \sum_{j=0}^\infty \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the $\{\varphi_j\}_{j=0}^\infty$ form a *dyadic partition of unity*. Given any $f \in S'(\mathbb{R}^n)$, we denote by $\mathscr{F}f$ and $\mathscr{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively.

DEFINITION 1.3. Let $s \in \mathbb{R}$, $0 < q \leqslant \infty$, and let $\{\varphi_j\}$ be a dyadic partition of unity.

(i) Let $0<p\leqslant\infty$, $b\in\mathbb{R}$. The space $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that

$$\|f\mid B^{(s,b)}_{p,q}(\mathbb{R}^n)\|=\left(\sum_{j=0}^{\infty}2^{jsq}(1+j)^{bq}\,\|\mathscr{F}^{-1}\varphi_j\mathscr{F}f\mid L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{1.4}$$

(with the usual modification if $q=\infty$) is finite.

(ii) Let $0<p<\infty$. The space $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that

$$\|f\mid F^s_{p,q}(\mathbb{R}^n)\|=\left\|\left(\sum_{j=0}^{\infty}2^{jsq}\,|\mathscr{F}^{-1}\varphi_j\mathscr{F}f(\cdot)|^q\right)^{1/q}\mid L_p(\mathbb{R}^n)\right\| \tag{1.5}$$

(with the usual modification if $q=\infty$) is finite.

When $b=0$, part (i) coincides with the usual definition for B-spaces, $B^{(s,0)}_{p,q}=B^s_{p,q}$, see [33, Definition 2.3.1/2, p. 45].

Remark 1.4. The theory of the spaces $B^s_{p,q}$ $(b=0)$ and $F^s_{p,q}$ has been developed in detail in [33, 34]. Recall that these two scales $B^s_{p,q}$ and $F^s_{p,q}$ cover (fractional) Sobolev spaces, Hölder–Zygmund spaces, local Hardy spaces, and classical Besov spaces—characterised via derivatives and differences.

We give a very useful characterisation of spaces $\mathrm{Lip}^{(1,-\alpha)}$ in terms of Zygmund spaces $\mathscr{C}^s=B^s_{\infty,\infty}$ at this point. This result was recently proved by Krbec and Schmeisser in [19, Proposition 2.5]. As we shall often apply it in the sequel, it is convenient to recall it here.

Proposition 1.5 (Krbec and Schmeisser [19, Proposition 2.5]). *Let* $\alpha>0$. *Then* $f\in\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ *if, and only if,* f *belongs to* $C(\mathbb{R}^n)$ *and there is some* $c>0$ *such that for all* λ, $0<\lambda<1$,

$$\|f\mid\mathscr{C}^{1-\lambda}(\mathbb{R}^n)\|\leqslant c\lambda^{-\alpha}$$

Moreover,

$$\sup_{0<\lambda<1}\ \lambda^{\alpha}\,\|f\mid\mathscr{C}^{1-\lambda}(\mathbb{R}^n)\| \tag{1.6}$$

is an equivalent norm in $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$.

This theorem is obviously of extrapolation type; for more details about extrapolation techniques we refer to [25]. Note that the spaces $\mathscr{C}^{1-\lambda}$ are defined via first differences here which requires some care later on when $\lambda\downarrow 0$.

When Ω is a bounded domain in $\mathbb{R}^n$, we introduce spaces $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$, $\alpha \geqslant 0$, by the usual adaption of Definition 1.1, i.e., those $f \in C(\bar{\Omega})$ such that

$$\|f \mid \mathrm{Lip}^{(1,-\alpha)}(\Omega)\| = \|f \mid L_\infty(\Omega)\| + \sup_{\substack{x,\, x+h \in \Omega \\ 0<|h|<1/2}} \frac{|(\Delta_h f)(x)|}{|h|\, |\log |h||^\alpha} \tag{1.7}$$

is finite. Note that by a bounded C^∞ domain in $\mathbb{R}^n$ we always have in mind the sense of [11, Definition V.4.1, p. 244] in the sequel. Standard procedures (see, for example, [11, pp. 250-251]) show that there is a bounded extension map from $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ to $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$.

Remark 1.6. The spaces $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ can also be obtained as a special case of the more general spaces $C^{0,\sigma(t)}(\bar{\Omega})$ which were introduced by Kufner, John, and Fučik; see [20, Definition 7.2.12, p. 361].

The spaces $B^{(s,b)}_{p,q}(\Omega)$ are defined by restriction, as usual, see [22, p. 8; 13, Definition 1.11]. For simplicity we shall mainly assume $\Omega = U = \{x \in \mathbb{R}^n : |x| < 1\}$ throughout this paper, i.e., that Ω is the unit ball in $\mathbb{R}^n$. One can easily check that our results remain true when U is replaced by some arbitrary bounded C^∞ domain $\Omega \subset \mathbb{R}^n$ (in the above sense), but at the expense of some constants (depending on Ω).

2. EMBEDDINGS

Here we study embeddings between spaces of (logarithmic) Lipschitz (and Zygmund) type, and Besov- or Triebel–Lizorkin-type, $B^s_{p,q}$ and $F^s_{p,q}$, respectively. Their definitions can be found in the previous section. We do not aim at completeness (of all possible embeddings) at all, but collect what is known so far in this area, complemented by some new results. The well-known forerunner of assertions of this type is certainly the celebrated result of Brézis and Wainger [3] in which it was shown that every function u in $H^{1+n/p}_p(\mathbb{R}^n)$ is "almost" Lipschitz-continuous, in the sense that for all, $x, y \in \mathbb{R}^n$, $x \neq y$, $|x-y| < 1/2$,

$$|u(x) - u(y)| \leqslant c\, |x-y|\, |\log |x-y||^{1/p'}\, \|u \mid H^{1+n/p}_p(\mathbb{R}^n)\|.$$

Here c is a constant independent of x, y and u, and $1/p' + 1/p = 1$. In our notation this simply means that $H^{1+n/p}_p(\mathbb{R}^n)$ is continuously embedded in $\mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n)$, that is,

$$H^{1+n/p}_p(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n). \tag{2.1}$$

This was our essential motivation in [13] to investigate whether the embedding (2.1) is *sharp* in the sense that

$$H_p^{1+n/p}(\mathbb{R}^n) \not\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$$

if $\alpha < 1/p'$. Moreover, turning to spaces defined on bounded domains, it then becomes reasonable to ask for which parameters embeddings of the above type (2.1) (suitably adapted to function spaces on domains) become compact. This immediately leads to the study of entropy numbers (and approximation numbers), but this subject is postponed to Section 3. We concentrate on embeddings first.

2.1. *Sharp Embeddings*

We start with some generalisation and refinement of the Brézis–Wainger result (2.1); see Section 1 for all necessary definitions.

THEOREM 2.1. *Let* $0 < p \leqslant \infty$ ($p < \infty$ *in* F-*case*), $0 < q \leqslant \infty$, *and* $\alpha \geqslant 0$. *Then*

$$B_{p,q}^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \qquad \textit{if, and only if,} \quad \alpha \geqslant \frac{1}{q'}, \tag{2.2}$$

and

$$F_{p,q}^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \qquad \textit{if, and only if,} \quad \alpha \geqslant \frac{1}{p'}. \tag{2.3}$$

Proof. When $1 < q \leqslant \infty$ (in B-case) or $1 < p < \infty$ (in F-case), our previous result [13, Theorem 2.1] implies the above assertions. Moreover, for $0 < q \leqslant 1$ (in the B-case) and $0 < p \leqslant 1$ (in the F-case) the sufficiency is covered by [15, (2.3.3/9,10), p. 45], respectively. ∎

Remark 2.2. We proved our result [13, Theorem 2.1] using (sub-)-atomic decompositions of function spaces, interpolation arguments, and extremal functions. Another way to prove (2.2) when $p = \infty$ and $1 \leqslant q \leqslant \infty$ (apart from the sharpness assertion) is given by Marchaud's inequality. One uses equivalent characterisations of $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $B_{p,q}^s(\mathbb{R}^n)$, via the modulus of continuity: cf. [2, Chap. 5, Sect. 4, pp. 332–334; 10, Chap. 2, Sects. 7–10, pp. 44–56] for details. Moreover, we use a similar argument in the proof of Proposition 4.2(ii) below; thus we refer to this point for some more explanation.

Remark 2.3. Recall $F_{p,2}^s = H_p^s$, $s \in \mathbb{R}$, $1 < p < \infty$. Thus we regain by (2.3) the Brézis–Wainger result (2.1), see [3, 14]. The sharpness result seems new (see also [12]).

Dealing with logarithmic Besov spaces $B^{(s,b)}_{p,q}$ instead of the Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}$ as above, Leopold obtained in [22, Theorem 1] results closely linked to Theorem 2.1. This naturally led us to study the interplay between spaces of type $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ (defined in the Fourier-analytical way) and spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\mathscr{C}^{(1,-\alpha)}(\mathbb{R}^n)$ (defined via differences) in general. Recall that for $\alpha=0$ it is known that $\mathscr{C}^s=B^s_{\infty,\infty}$, $s>0$, see [33, Theorem 2.5.7(ii), p. 90], and $B^1_{\infty,1}\hookrightarrow \mathrm{Lip}\hookrightarrow B^1_{\infty,\infty}$, see [33, (2.5.7/2), (2.5.7/11), pp. 89–90]. In [13, Propositions 4.2, 4.4] we proved that there are extensions to $\alpha\geqslant 0$ in the following sense,

$$B^{(1,-\alpha)}_{\infty,1}(\mathbb{R}^n)\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\hookrightarrow \mathscr{C}^{(1,-\alpha)}(\mathbb{R}^n)=B^{(1,-\alpha)}_{\infty,\infty}(\mathbb{R}^n). \tag{2.4}$$

Moreover, one has $B^{(1,-\alpha)}_{\infty,q}(\mathbb{R}^n)\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ if, and only if, $0<q\leqslant 1$. Note that this is well known for $\alpha=0$, see [15, (2.3.3/9, 10), p. 45]. Having thus established some assertion about spaces of type $B^{(s,b)}_{p,q}$ related to spaces $\mathrm{Lip}^{(1,-\alpha)}$, $\mathscr{C}^{(1,-\alpha)}$, we now study in detail (sharp) embeddings between the latter two, i.e., between logarithmic Lipschitz and Zygmund spaces, both of which are defined by differences.

PROPOSITION 2.4. *Let α, β, γ be non-negative real numbers. Then*

$$\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\hookrightarrow \mathscr{C}^{(1,-\beta)}(\mathbb{R}^n)\hookrightarrow \mathrm{Lip}^{(1,-\gamma)}(\mathbb{R}^n) \tag{2.5}$$

if, and only if,

$$\beta\geqslant\alpha, \qquad \textit{and} \qquad \gamma\geqslant\beta+1.$$

Proof. *Step* 1. As all spaces are defined on $\mathbb{R}^n$ we omit all reference to $\mathbb{R}^n$ in what follows. We start with the left-hand embedding in (2.5), i.e., we show

$$\mathrm{Lip}^{(1,-\alpha)}\hookrightarrow \mathscr{C}^{(1,-\beta)} \qquad \text{if, and only if,} \quad \beta\geqslant\alpha. \tag{2.6}$$

The sufficiency is covered by (2.4) and an obvious monotonicity argument. To prove the necessity, we proceed by contradiction. Assume that $\mathrm{Lip}^{(1,-\alpha)}\hookrightarrow \mathscr{C}^{(1,-\beta)}$ for some $\beta<\alpha$. Another application of (2.4) thus implies that

$$B^{(1,-\alpha)}_{\infty,1}\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}\hookrightarrow \mathscr{C}^{(1,-\beta)}=B^{(1,-\beta)}_{\infty,\infty} \tag{2.7}$$

for some $\beta<\alpha$; in particular, $B^{(1,-\alpha)}_{\infty,1}\hookrightarrow B^{(1,-\beta)}_{\infty,\infty}$ for some $\beta<\alpha$. But this contradicts Leopold's related result [22, Theorem 1]. Hence (2.6) is proved.

Step 2. We deal with the sufficiency for the right-hand embedding in (2.5), that is,

$$\mathscr{C}^{(1,\,-\beta)} \hookrightarrow \mathrm{Lip}^{(1,\,-\gamma)} \qquad \text{if} \quad \gamma \geqslant \beta + 1. \tag{2.8}$$

Note that the case $\beta = 0$ is covered by Theorem 2.1(ii), see also [13, Remark 2.3]. Moreover, via the embeddings

$$\mathscr{C}^{(1,\,-\beta)} = B^{(1,\,-\beta)}_{\infty,\,\infty} \hookrightarrow B^{(1,\,-\gamma)}_{\infty,\,1} \hookrightarrow \mathrm{Lip}^{(1,\,-\gamma)}$$

if $\gamma > \beta + 1$, we need to prove (2.8) for $\gamma = \beta + 1$ only; see [22, Theorem 1] and (2.4). Consequently we have to show

$$\mathscr{C}^{(1,\,-\beta)} \hookrightarrow \mathrm{Lip}^{(1,\,-(\beta+1))}, \qquad \beta > 0,$$

that is,

$$\| f \mid \mathrm{Lip}^{(1,\,-(\beta+1))} \| \leqslant c\, \| f \mid \mathscr{C}^{(1,\,-\beta)} \|, \qquad \beta > 0, \tag{2.9}$$

for some $c > 0$ and all $f \in \mathscr{C}^{(1,\,-\beta)}$. Let $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth partition of unity. We obtained in [13, (4.5)] that

$$\| f \mid \mathrm{Lip}^{(1,\,-(\beta+1))} \| \leqslant c \Bigg(\| f \mid L_\infty \| + \sup_{k \in \mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=0}^{k} 2^{j-k} \, \| (\varphi_j \hat{f})^\vee \mid L_\infty \| \\ + \sup_{k \in \mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=k+1}^{\infty} \| (\varphi_j \hat{f})^\vee \mid L_\infty \| \Bigg), \tag{2.10}$$

and likewise,

$$\| f \mid \mathscr{C}^{(1,\,-\beta)} \| \sim \| f \mid L_\infty \| + \sup_{j \in \mathbb{N}_0} 2^j (1+j)^{-\beta} \, \| (\varphi_j \hat{f})^\vee \mid L_\infty \|, \tag{2.11}$$

see [13, (4.8)] and Step 2 of the proof of [13, Proposition 4.2]. In view of (2.9)–(2.11) it is thus sufficient to prove

$$\sup_{k \in \mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=0}^{k} 2^{j-k} \, \| (\varphi_j \hat{f})^\vee \mid L_\infty \| \\ \leqslant c \sup_{j \in \mathbb{N}_0} 2^j (1+j)^{-\beta} \, \| (\varphi_j \hat{f})^\vee \mid L_\infty) \| \tag{2.12}$$

and

$$\sup_{k\in\mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=k+1}^{\infty} \|(\varphi_j\hat{f})^\vee \mid L_\infty\| \leqslant c' \sup_{j\in\mathbb{N}_0} 2^j(1+j)^{-\beta} \|(\varphi_j\hat{f})^\vee \mid L_\infty\| \tag{2.13}$$

in order to prove (2.8). We start with (2.12) and obtain for $\beta>0$,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=0}^{k} 2^{j-k} \|(\varphi_j\hat{f})^\vee \mid L^\infty\| \\
&\quad= \sup_{k\in\mathbb{N}} k^{-(\beta+1)} \sum_{j=0}^{k} 2^j \|(\varphi_j\hat{f})^\vee \mid L_\infty\| \\
&\quad\leqslant \sup_{j\in\mathbb{N}_0} 2^j(1+j)^{-\beta} \|(\varphi_j\hat{f})^\vee \mid L_\infty\| \sup_{k\in\mathbb{N}} k^{-(\beta+1)} \sum_{j=0}^{k} (1+j)^\beta \\
&\quad\leqslant c \sup_{j\in\mathbb{N}_0} 2^j(1+j)^{-\beta} \|(\varphi_j\hat{f})^\vee \mid L_\infty\|.
\end{aligned}$$

We handle (2.13) and conclude

$$\begin{aligned}
\sup_{k\in\mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=k+1}^{\infty} \|(\varphi_j\hat{f})^\vee \mid L_\infty\| &\leqslant c \sup_{j\in\mathbb{N}_0} 2^j(1+j)^{-\beta} \|(\varphi_j\hat{f})^\vee \mid L_\infty\| \\
&\quad\times \sup_{k\in\mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=k+1}^{\infty} 2^{-j}(1+j)^\beta
\end{aligned}$$

Note that $\sup_{k\in\mathbb{N}} 2^k k^{-(\beta+1)} \sum_{j=k+1}^{\infty} 2^{-j}(1+j)^\beta \leqslant c'$, and hence (2.13) is proved.

Step 3. It remains to show the necessity of $\gamma\geqslant\beta+1$ in order to establish the embedding $\mathscr{C}^{(1,-\beta)} \hookrightarrow \mathrm{Lip}^{(1,-\gamma)}$. Note that the sharpness result in Theorem 2.1(ii) (with $p=q=\infty$) covers the case $\beta=0$. Now assume $\gamma<\beta+1$. There is some number v such that $\gamma<v<\beta+1$. We prove that there are functions $f_{\varepsilon,v}(x)$, such that $f_{\varepsilon,v}\notin \mathrm{Lip}^{(1,-\gamma)}$, and $f_{\varepsilon,v}\in\mathscr{C}^{(1,-\beta)}$, where $\beta>0$ will be chosen sufficiently small. Put

$$f_{\varepsilon,v}(x)=\begin{cases} 0, & x=0,\\ |x|\,|\log|x||^v, & 0<|x|\leqslant\varepsilon,\\ \varepsilon\,|\log\varepsilon|^v, & |x|>\varepsilon.\end{cases}$$

Clearly, $\|f_{\varepsilon,\nu} \mid L_\infty\| = \varepsilon\,|\log \varepsilon|^\nu$. We estimate $\|f_{\varepsilon,\nu} \mid \mathrm{Lip}^{(1,-\gamma)}\|$. One obtains for $0<|h|<1/2$ that

$$\frac{|\Delta_h f_{\varepsilon,\nu}(0)|}{|h|\,|\log|h||^\gamma} = \begin{cases} \dfrac{\varepsilon\,|\log\varepsilon|^\nu}{|h|\,|\log|h||^\gamma}, & \varepsilon \leqslant |h| < 1/2, \\ |\log|h||^{\nu-\gamma}, & |h|<\varepsilon, \end{cases}$$

where $\varepsilon>0$ is chosen sufficiently small. Now

$$\sup_{\substack{x\in\mathbb{R}^n\\0<|h|<1/2}} \frac{|\Delta_h f_{\varepsilon,\nu}(x)|}{|h|\,|\log|h||^\gamma} \geqslant \sup_{0<|h|<\varepsilon} \frac{|\Delta_h f_{\varepsilon,\nu}(0)|}{|h|\,|\log|h||^\gamma} = \sup_{0<|h|<\varepsilon} |\log|h||^{\nu-\gamma},$$

which is not bounded from above for $\nu>\gamma$. Thus $f_{\varepsilon,\nu}\notin \mathrm{Lip}^{(1,-\gamma)}$. It remains to show $f_{\varepsilon,\nu}\in\mathscr{C}^{(1,-\beta)}$. Note that

$$\sup_{\substack{x\in\mathbb{R}^n\\0<|h|<1/2}} \frac{|\Delta_h^2 f_{\varepsilon,\nu}(x)|}{|h|\,|\log|h||^\beta} \leqslant c_1 \sup_{0<|h|<1/2} \frac{|\Delta_h^2 f_{\varepsilon,\nu}(0)|}{|h|\,|\log|h||^\beta} + c_2.$$

Straightforward calculation yields for $\nu<\beta+1$ that $\sup_{0<|h|<1/2} (|\Delta_h^2 f_{\varepsilon,\nu}(0)|/(|h|\,|\log|h||^\beta)) \leqslant c\,|\log\varepsilon|^{\nu-\beta}$ so that finally

$$\|f_{\varepsilon,\nu} \mid \mathscr{C}^{(1,-\beta)}\| \leqslant c(\varepsilon\,|\log\varepsilon|^\nu + |\log\varepsilon|^{\nu-\beta}) \leqslant c'\,|\log\varepsilon|^{\nu-\beta}$$

when $\nu<\beta+1$. Hence there are functions $\{f_{\varepsilon,\nu}\}_{0<\varepsilon\leqslant\varepsilon_0}$, with $\gamma<\nu<\beta+1$, $\varepsilon_0>0$ sufficiently small, belonging to $\mathscr{C}^{(1,-\beta)}\backslash\mathrm{Lip}^{(1,-\gamma)}$. This ends the proof. ∎

2.2. *Compact Embeddings*

Recall that $U=\{x\in\mathbb{R}^n : |x|<1\}$ is the unit ball in $\mathbb{R}^n$. In this section we give a few results concerning compact embeddings of the type studied above when the corresponding function spaces are defined on a bounded domain. We intend to give some standard situations only, and do not aim at completeness.

PROPOSITION 2.5. *Let* $0<p,\ q\leqslant\infty$, $\alpha>1/q'$. *Then* id^B: $B^{1+n/p}_{p,q}(U)\to \mathrm{Lip}^{(1,-\alpha)}(U)$ *is compact.*

The above corollary is a consequence of [13, Theorem 3.5] where we moreover estimated the corresponding entropy numbers of id^B from above. It can also be obtained using Theorem 2.1 and Proposition 2.6 below.

PROPOSITION 2.6. *Let $\beta > \alpha > 0$. Then $id_{\alpha\beta}$: $\mathrm{Lip}^{(1,-\alpha)}(U) \to \mathrm{Lip}^{(1,-\beta)}(U)$ is compact.*

Proof. We make use of characterisation (1.6) as well as of the compactness of the embedding

$$id: \mathscr{C}^{1-\lambda}(U) \to \mathscr{C}^{1-\mu}(U), \qquad \lambda < \mu, \tag{2.14}$$

see [15, Sect. 2.5.1], for instance. Let

$$U_\alpha = \{f \in \mathrm{Lip}^{(1,-\alpha)}(U) : \|f \mid \mathrm{Lip}^{(1,-\alpha)}(U)\| < 1\} \tag{2.15}$$

be the unit ball in $\mathrm{Lip}^{(1,-\alpha)}(U)$ and let $\varepsilon > 0$. We construct a finite ε-net for U_α in $\mathrm{Lip}^{(1,-\beta)}(U)$. We know by (1.6) that for every $f \in U_\alpha$ we have $\lambda^\alpha f \in \mathscr{C}^{1-\lambda}(U)$, $\|\lambda^\alpha f \mid \mathscr{C}^{1-\lambda}(U)\| \leqslant c$ for some $c > 0$ and any number λ, $0 < \lambda < 1$. Without restriction of generality we may assume that $c < 1$, that is, $\lambda^\alpha f$ belongs to the unit ball in $\mathscr{C}^{1-\lambda}(U)$. Now (2.14) implies that in any space $\mathscr{C}^{1-\mu}(U)$, $\mu > \lambda$, there is a finite ε-net for the unit ball in $\mathscr{C}^{1-\lambda}(U)$ as $\mu > \lambda$. We put $\mu := \lambda^{\alpha/\beta}$; hence $\mu \in (0, 1)$ and $\mu > \lambda$ by $\lambda \in (0, 1)$ and $\alpha < \beta$. Let $\{h_j\}_{j=1}^N \in \mathscr{C}^{1-\mu}(U)$ be some finite $\varepsilon/2$-net for $\lambda^\alpha U_\alpha \subset \mathscr{C}^{1-\lambda}(U)$. Then for any $f \in U_\alpha$ there is some $j \in \{1, ..., N\}$ such that $\|\lambda^\alpha f - h_j \mid \mathscr{C}^{1-\mu}(U)\| < \varepsilon/2$. Put $g_j := \mu^{-\beta} h_j$, $j = 1, ..., N$, and note that $\lambda^\alpha = \mu^\beta$. Thus $\mu^\beta \|f - g_j \mid \mathscr{C}^{1-\mu}(U)\| < \varepsilon/2$, and taking the supremum over all $\mu \in (0, 1)$ we obtain by (1.6) that $\|f - g_j \mid \mathrm{Lip}^{(1,-\beta)}(U)\| \leqslant \varepsilon/2 < \varepsilon$. Furthermore, $f \in \mathrm{Lip}^{(1,-\alpha)}(U) \hookrightarrow \mathrm{Lip}^{(1,-\beta)}(U)$ for $\beta \geqslant \alpha$ implies that $g_j \in \mathrm{Lip}^{(1,-\beta)}(U)$. ▮

Remark 2.7. In Remark 1.6 we identified $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ as a special case of the more general $C^{0,\sigma(t)}(\bar{\Omega})$ spaces introduced in [20, Definition 7.2.12, p. 361]. The above proposition can also be found as a special case of a related result for $C^{0,\sigma(t)}(\bar{\Omega})$ spaces, that is [20, Lemma 7.4.3, p. 368].

COROLLARY 2.8. *Let $0 < p < \infty, 0 < q \leqslant \infty, \alpha > 1/p'$. Then $id^F : F^{1+n/p}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)$ is compact.*

Proof. This follows immediately from Theorem 2.1 and Proposition 2.6. ▮

Leopold obtained similar results in [22, Theorem 2] when dealing with logarithmic Besov spaces $B^{(s,b)}_{p,q}$ exclusively, see also [13, Proposition 4.7].

As we already mentioned at the beginning of this section, we have dealt with some model cases only. However, more compactness results can be easily obtained from our results in Subsections 3.2 and 3.3 below when we deal with estimates for entropy numbers and approximation numbers (of compact embeddings).

3. ENTROPY NUMBERS AND APPROXIMATION NUMBERS

We briefly recall the definitions of entropy and approximation numbers. Let A_1 and A_2 be two complex quasi-Banach spaces and let T be a linear and continuous operator from A_1 into A_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in A_2 of radius ε which cover the image TU_1 of the unit ball $U_1 = \{a \in A_1 : \|a \mid A_1\| \leqslant 1\}$.

DEFINITION 3.1. Let $k \in \mathbb{N}$ and let $T: A_1 \to A_2$ be the above continuous operator.

(i) The *kth entropy number* e_k of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover TU_1.

(ii) The *kth approximation number* a_k of T is the infimum of all numbers $\|T - L\|$ where L runs through the collection of all continuous linear maps from A_1 to A_2 with rank $L < k$.

For details and properties of these numbers (like additivity and multiplicativity, for instance) we refer to [7, 11, 18, 28] (always restricted to the case of Banach spaces). The extension of these properties to quasi-Banach spaces causes no problems, see [15].

3.1. ℓ_p-*Spaces*

As in [13], our estimation of the entropy numbers of embedding maps involves a reduction of the problem to the study of maps between finite-dimensional sequence spaces; this method has been efficiently used before in [15, 35]. Accordingly we introduce the sequence spaces ℓ_p^M, $M \in \mathbb{N}$, $0 < p \leqslant \infty$ and follow [15, 3.21, p. 97]. By ℓ_p^M we shall mean the linear space of all complex M-tuples $y = (y_j)$, endowed with the quasi-norm

$$\|y \mid \ell_p^M\| = \left(\sum_{j=1}^{M} |y_j|^p \right)^{1/p}, \qquad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Moreover, we also need weighted ℓ_p-spaces in the following sense: Let $(M_j)_{j \in \mathbb{N}_0}$ be a sequence of natural numbers with $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. Let $(w_j)_{j \in \mathbb{N}_0}$ be a sequence of positive numbers (weights), mainly of the type

$$w_j = 2^{j\delta} \qquad \text{or} \qquad w_j = \langle j \rangle^{\varkappa}, \qquad j \in \mathbb{N}_0, \quad \delta > 0, \quad \varkappa \in \mathbb{R}.$$

In [13] we slightly extended the definition of Triebel given in [35, 8.1, p. 38] in the following sense. Let $\ell_q(w_j \ell_p^{M_j})$ be the linear space of all complex sequences $x=(x_{j,l}: j\in\mathbb{N}_0;\ l=1,...,M_j)$ endowed with the quasi-norm

$$\|x \mid \ell_q(w_j \ell_p^{M_j})\| = \left(\sum_{j=0}^{\infty} w_j^q \left(\sum_{l=1}^{M_j} |x_{j,l}|^p \right)^{q/p} \right)^{1/q} \tag{3.1}$$

(with the obvious modifications if $p=\infty$ or $q=\infty$). In case of $w_j \equiv 1$ we write $\ell_q(\ell_p^{M_j})$. The above notation was introduced in [13, (3.1)] and coincides with [35, (8.2), p. 38] when $w_j = 2^{j\delta}$, $\delta>0$.

Concerning entropy numbers of the embedding map $id: \ell_{p_1}^M \to \ell_{p_2}^M$, $0<p_1\leqslant p_2 \leqslant \infty$, we make use of the results [15, Proposition 3.2.2, p. 98; 35, Proposition 7.2, p. 36]. Note that in the Banach space setting estimates for the entropy numbers in finite-dimensional sequence spaces, see, for example, (3.21) below, have been studied in great detail for a long time. We refer to [30] as well as [18, Sect. 3.c.8] for further details and references. In case of approximation numbers (in the same setting) we mention the paper by Gluskin [17], the survey article by R. Linde [24], the extension to quasi-Banach space setting by Edmunds and Triebel in [15, Sect. 3.2.3] and the recent paper of Caetano [4].

We consider the embedding

$$id_{p_1,p_2}: \ell_q(\ell_{p_1}^{M_j}) \to \ell_q(\langle j\rangle^{-\varkappa}\, \ell_{p_2}^{M_j}), \tag{3.2}$$

where $0<p_1\leqslant p_2\leqslant\infty$, $0<q\leqslant\infty$, $\varkappa>0$, and $M_j \sim 2^{jn}$, $j\in\mathbb{N}_0$. We have shown in [13, Proposition 3.1] that id_{p_1,p_2} is compact for $\varkappa>0$ and $p_1\leqslant p_2$ (see also Proposition 3.2 below which implies the compactness, too). We study (the asymptotic behaviour of) the corresponding entropy numbers $e_k(id_{p_1,p_2})$ and approximation numbers $a_k(id_{p_1,p_2})$ in the sequel. Note that in case of entropy numbers parallel results—i.e., when dealing with dyadic weights of type $w_j=2^{j\delta}$, $\delta>0$—were obtained by Kühn in [21] and Triebel in [35, Sect. 8].

It turns out that for later application we need only deal with the cases when $p=p_1=p_2$ and $p=p_1$, $p_2=\infty$, respectively. We begin with the setting when $0<p=p_1=p_2\leqslant\infty$ and adopt the notation

$$id_{p,p}: \ell_q(\ell_p^{M_j}) \to \ell_q(\langle j\rangle^{-\varkappa}\, \ell_p^{M_j}), \tag{3.3}$$

where $0<p\leqslant\infty$, $0<q\leqslant\infty$, $\varkappa>0$, and $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$. As a first result we obtained in [13] the following.

Proposition 3.2 [13, Proposition 3.1]. *Let* $\varkappa>0$, $0<p\leqslant\infty$, $0<q\leqslant\infty$, $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$. *Then*

$$e_k(id_{p,p}) \sim (\log\langle k\rangle)^{-\varkappa}, \qquad k\in\mathbb{N}.$$

We give the counterpart of Proposition 3.2 in terms of approximation numbers. Recall our notation (3.3).

PROPOSITION 3.3. *Let* $\varkappa > 0$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. *Then*

$$a_k(id_{p,\,p}) \sim (\log \langle k \rangle)^{-\varkappa}, \qquad k \in \mathbb{N}. \tag{3.4}$$

Proof. The proof is essentially based on our proof of Proposition 3.1 in [13]. Recall our decomposition of $id_{p,\,p}$ into

$$id_{p,\,p} = \sum_{j=0}^{\infty} id_j, \tag{3.5}$$

where

$$\begin{aligned} id_j x &= (\delta_{jk} x_{k,\,l} \colon k \in \mathbb{N}_0,\, l = 1, ..., M_k) \\ &= (0, ..., 0, x_{j,\,1}, ..., x_{j,\,M_j}, 0, ...). \end{aligned} \tag{3.6}$$

One obtains that

$$\| id_j x \mid \ell_q(\langle k \rangle^{-\varkappa}\, \ell_m^{M_k}) \| \leqslant \langle j \rangle^{-\varkappa}\, \| x \mid \ell_q(\ell_p^{M_k}) \|. \tag{3.7}$$

We make use of the following commutative diagram

$$\begin{array}{ccc} \ell_p^{M_j} & \xrightarrow{\quad id_j \quad} & \ell_q(\ell_p^{M_j}) \\ {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle id_{p,\,p}} \\ \ell_p^{M_j} & \xleftarrow{\quad id_j \quad} & \ell_q(\langle j \rangle)^{-\varkappa}\, \ell_p^{M_j}) \end{array} \tag{3.8}$$

where id_j is given by (3.6) (acting in the slightly modified way as indicated above) and id^j maps $\ell_p^{M_j}$ identically onto $\ell_q(\ell_p^{M_j})$ interpreted as dyadic blocks. Obviously,

$$id(\ell_p^{M_j} \to \ell_p^{M_j}) = id_j \circ id_{p,\,p} \circ id^j.$$

Note that $\| id_j \| = \langle j \rangle^{\varkappa}$ by (3.7), $\| id^j \| = 1$; thus the multiplicativity of approximation numbers yields

$$a_k(id_{p,\,p}) \geqslant c \langle j \rangle^{-\varkappa}\, a_k(id \colon \ell_p^{M_j} \to \ell_p^{M_j}). \tag{3.9}$$

We have $a_k(id \colon \ell_p^{M_j} \to \ell_p^{M_j}) = 1$ for $1 \leqslant k \leqslant M_j$; hence $a_{2^{jn}}(id_{p,\,p}) \geqslant c \langle j \rangle^{-\varkappa}$. This gives the estimate from below in (3.4).

For the upper estimate we adapt the proof in [13, Proposition 3.1] (dealing with entropy numbers) in a suitable manner. Let $J \in \mathbb{N}$. We split the sum in (3.5) into two parts,

$$id_{p,p} = \sum_{j=0}^{J} id_j + \sum_{j=J+1}^{\infty} id_j, \tag{3.10}$$

with $\|\sum_{j=J+1}^{\infty} id_j\| \leqslant c\langle J\rangle^{-\varkappa}$, and $a_k(id_j) = \langle j\rangle^{-\varkappa} a_k(id\colon \ell_p^{M_j} \to \ell_p^{M_j})$. Let $\varrho = \min(1, p, q)$; then the additivity of approximation numbers leads to

$$\begin{aligned} a_k^{\varrho}(id_{p,p}) &\leqslant c\left(\langle J\rangle^{-\varkappa\varrho} + \sum_{j=0}^{J} \langle j\rangle^{-\varkappa\varrho}\, a_{k_j}^{\varrho}(id\colon \ell_p^{M_j} \to \ell_p^{M_j})\right), \\ k &= \sum_{j=0}^{J} k_j. \end{aligned} \tag{3.11}$$

We put $k_j = 2M_j$, $j = 0, ..., J$; then $a_{k_j}(id\colon \ell_p^{M_j} \to \ell_p^{M_j}) = 0$, $j = 0, ..., J$, and $k = \sum_{j=0}^{J} k_j \sim 2^{Jn}$. Hence (3.11) implies $a_{c2^{Jn}}(id_{p,p}) \leqslant c\langle J\rangle^{-\varkappa}$, and the proof is finished. ∎

We study the embedding

$$id_{p,\infty}\colon \ell_q(\ell_p^{M_j}) \to \ell_q(\langle j\rangle^{-\varkappa}\, \ell_\infty^{M_j}) \tag{3.12}$$

now, where $0 < p < \infty$, $0 < q \leqslant \infty$ and $\varkappa > 0$. Note that the compactness of $id_{p,\infty}$ is a consequence of the compactness of $id_{p,p}$. Moreover, in view of Propositions 3.2, 3.3, we may assume $0 < p < \infty$ in the sequel. We shall estimate the corresponding entropy numbers and approximation numbers, beginning with entropy numbers.

Proposition 3.4. *Let $\varkappa > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. There is some $c > 0$ such that for all $k \in \mathbb{N}$,*

$$e_k(id_{p,\infty}) \geqslant c \begin{cases} k^{-1/p}(\log\langle k\rangle)^{-\varkappa}, & \varkappa > 1/p \\ k^{-\varkappa}, & \varkappa \leqslant 1/p. \end{cases} \tag{3.13}$$

Moreover, if we additionally have $1 \leqslant p < \infty$, then (3.13) can be replaced by

$$e_k(id_{p,\infty}) \geqslant c \begin{cases} k^{-1/p}(\log\langle k\rangle)^{-\varkappa+1/p}, & \varkappa > 1/p \\ k^{-\varkappa}, & \varkappa \leqslant 1/p. \end{cases} \tag{3.14}$$

Proof. We adapt our proof of Proposition 3.1 in [13], see Proposition 3.2, to the situation described above. Then diagram (3.8) has to be replaced by

$$\begin{array}{ccc} \ell_p^{M_j} & \xrightarrow{id^j} & \ell_q(\ell_p^{M_j}) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id_{p,\infty}} \\ \ell_\infty^{M_j} & \xleftarrow{id_j} & \ell_q(\langle j\rangle^{-\varkappa}\,\ell_\infty^{M_j}) \end{array} \tag{3.15}$$

see the proof of Proposition 3.3 for interpretation. Recall $\|id_j\| = \langle j\rangle^{\varkappa}$ by (3.7) and $\|id^j\| = 1$. Again, the multiplicativity of entropy numbers yields

$$e_k(id_{p,\infty}) \geqslant c\langle j\rangle^{-\varkappa}\, e_k(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}).$$

Put $k \sim 2M_j \sim 2^{jn}$; then we obtain by [35, Proposition 7.2]

$$e_{c2^{jn}}(id_{p,\infty}) \geqslant c'\langle j\rangle^{-\varkappa}\, e_{c2^{jn}}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \geqslant C\langle j\rangle^{-\varkappa}\, 2^{-jn/p},$$

leading to the upper line in (3.13). However, putting $k \sim \log(2M_j) \sim j$, the above-mentioned result [35, Proposition 7.2] gives

$$e_{cj}(id_{p,\infty}) \geqslant c'\langle j\rangle^{-\varkappa}\, e_{cj}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \geqslant C\langle j\rangle^{-\varkappa},$$

and so the second lines in (3.13) and (3.14) are verified. Moreover, if we assume $1 \leqslant p < \infty$, we choose $k \sim 2^{j(n-v)}$, $0 < v < n$, thus $\log(2M_j) < k < 2M_j \sim 2^{jn}$ and we apply the corresponding lower estimate for $e_k(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j})$ according to [30; 18, Sect. 3.c.8; 35, Remark 7.5]. We arrive at

$$\begin{aligned} e_k(id_{p,\infty}) &\geqslant c_1\langle j\rangle^{-\varkappa}\, e_k(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \\ &\geqslant c_2\langle j\rangle^{-\varkappa}\, k^{-1/p}[\log(c_3 2^{jn} k^{-1})]^{1/p} \geqslant c_4\langle j\rangle^{-\varkappa+1/p}\, k^{-1/p}. \end{aligned}$$

Note that $j \sim \log k$ and hence we finally obtain

$$e_m(id_{p,\infty}) \geqslant cm^{-1/p}(\log\langle m\rangle)^{-\varkappa+1/p}, \qquad m \in \mathbb{N},$$

when $1 \leqslant p < \infty$. ∎

Next we deal with corresponding estimates for the entropy numbers from above. Our result is the following.

PROPOSITION 3.5. *Let* $\varkappa > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$ *and* $\varrho := \min(q, 1)$. *There is some* $c > 0$ *such that for all* $k \in \mathbb{N}$,

$$e_k(id_{p,\infty}) \leqslant c \begin{cases} k^{-1/p}(\log \langle k \rangle)^{-\varkappa + 1/\varrho + 2/p}, & \varkappa > \dfrac{1}{\varrho} + \dfrac{2}{p} \\ k^{-1/p}(\log \langle k \rangle)^{1/\varrho + 1/p}, & \varkappa = \dfrac{1}{\varrho} + \dfrac{2}{p} \\ k^{-1/p - \varkappa/(1/\varrho + 2/p)}, & \varkappa < \dfrac{1}{\varrho} + \dfrac{2}{p}. \end{cases} \tag{3.16}$$

Proof. *Step* 1. We modify our proof of Proposition 3.1 in [13]. Recall our decomposition of $id_{p,\infty}$ in the sense of (3.5), where id_j is given by (3.6). Let $L, J \in \mathbb{N}$, $L > J$, which will be chosen later. Unlike the proofs of Propositions 3.2, 3.3, see (3.10), we split the sum in (3.5) into three parts,

$$id_{p,\infty} = \sum_{j=0}^{J} id_j + \sum_{j=J+1}^{L} id_j + \sum_{j=L+1}^{\infty} id_j, \tag{3.17}$$

and obtain in the same way as in [13, (3.7)],

$$\left\| \sum_{j=L+1}^{\infty} id_j \right\| \leqslant c L^{-\varkappa}. \tag{3.18}$$

The additivity of entropy numbers thus leads for $\varrho = \min(1, q)$ to

$$\begin{aligned} e_k^{\varrho}(id_{p,\infty}) &\leqslant c \left(L^{-\varkappa\varrho} + \sum_{j=0}^{J} e_{k_j}^{\varrho}(id_j) + \sum_{j=J+1}^{L} e_{k_j}^{\varrho}(id_j) \right), \\ k &= \sum_{j=0}^{L} k_j. \end{aligned} \tag{3.19}$$

Recall

$$e_{k_j}(id_j) = \langle j \rangle^{-\varkappa} e_{k_j}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \tag{3.20}$$

by (3.15), and

$$e_{k_j}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \leqslant c \begin{cases} [k_j^{-1} \log(c 2^{jn} k_j^{-1})]^{1/p}, & k_j \leqslant 2M_j \\ 2^{-k_j/2M_j} 2^{-j(n/p)}, & k_j > 2M_j, \end{cases} \tag{3.21}$$

see [30; 18, Sect. 3.c.8; 35, Theorem 7.3]. Let $j = 0, ..., J$. We put $k_j = 2^{Jn} 2^{-(J-j)\varepsilon}$, $\varepsilon > 0$, such that

$$k_j = 2M_j 2^{(J-j)(n-\varepsilon)} > 2M_j, \qquad j = 0, ..., J,$$

if $\varepsilon > 0$ is small. Then (3.20), (3.21) imply that

$$e_{k_j}(id_j) \leqslant c \langle j \rangle^{-\varkappa} 2^{-j(n/p)} 2^{-2^{(J-j)(n-\varepsilon)}}$$
$$= c 2^{-J(n/p)} \langle J \rangle^{-\chi} 2^{(J-j)(n/p) - 2^{(J-j)(n-\varepsilon)}} \left(\frac{\langle J \rangle}{\langle j \rangle} \right)^{\varkappa}$$

such that

$$\sum_{j=0}^{J} e_{k_j}^{\varrho}(id_j) \leqslant c 2^{-J(n/p)\varrho} \langle J \rangle^{-\varkappa\varrho}, \qquad \sum_{j=0}^{J} k_j \leqslant c' 2^{Jn}.$$

Together with (3.19) we are thus led to

$$e_k^{\varrho}(id_{p,\infty}) \leqslant c \left(L^{-\varkappa\varrho} + 2^{-J(n/p)\varrho} \langle J \rangle^{-\varkappa\varrho} + \sum_{j=J+1}^{L} \langle j \rangle^{-\varkappa\varrho} e_{k_j}^{\varrho}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \right), \tag{3.22}$$

with $k \leqslant c 2^{Jn} + \sum_{j=J+1}^{L} k_j$.

Step 2. It remains to handle the sum $\sum_{j=J+1}^{L} \langle j \rangle^{-\varkappa\varrho} e_{k_j}^{\varrho}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j})$ in dependence upon the number $\varkappa$. First let $\varkappa > \frac{1}{\varrho} + \frac{2}{p}$. Thus we may find some number $v > 1$ such that $\varkappa > \frac{1}{\varrho} + \frac{v+1}{p} > \frac{1}{\varrho} + \frac{2}{p}$. Put $k_j = 2^{Jn} J^{v-1} j^{-v}$, $j = J+1, ..., L \sim 2^{Jn/v} J^{1-1/v}$. Then $k_j \leqslant 2M_j \sim 2^{jn}$ and we get

$$k \leqslant c 2^{Jn} + \sum_{j=J+1}^{L} k_j = c 2^{Jn} + 2^{Jn} J^{v-1} \sum_{j=J+1}^{L} j^{-v} \leqslant c' 2^{Jn}. \tag{3.23}$$

We apply (3.21) and obtain

$$e_{k_j}(id_j) \leqslant c \langle j \rangle^{-\varkappa} 2^{-J(n/p)} J^{-(v-1)/p} j^{v/p} (j-J)^{1/p}$$
$$\leqslant c' 2^{-J(n/p)} J^{-(v-1)/p} j^{-\varkappa + (v+1)/p}.$$

Consequently,

$$\sum_{j=J+1}^{L} e_{k_j}^{\varrho}(id_j) \leqslant c 2^{-J(n/p)\varrho} J^{-((v-1)/p)\varrho} \sum_{j=J+1}^{L} j^{-(\varkappa - (v+1)/p)\varrho}$$
$$\leqslant c' 2^{-J(n/p)\varrho} J^{(-\varkappa + 1/\varrho + 2/p)\varrho},$$

and together with (3.22), (3.23) we arrive at

$$e_{c_1 2^{Jn}}^{\varrho}(id_{p,\infty}) \leqslant c_2 (L^{-\varkappa\varrho} + 2^{-J(n/p)\varrho} J^{-\varkappa\varrho} + 2^{-J(n/p)\varrho} J^{(-\varkappa + 1/\varrho + 2/p)\varrho})$$
$$\leqslant c_3 (L^{-\varkappa\varrho} + 2^{-J(n/p)\varrho} J^{\varrho(-\varkappa + 1/\varrho + 2/p)})$$
$$\leqslant c_4 2^{-J(n/p)\varrho} J^{\varrho(-\varkappa + 1/\varrho + 2/p)}$$

when $L > J$ is sufficiently large. We have thus verified the first line in (3.16). Assume now $\varkappa + \frac{1}{\varrho} + \frac{2}{p}$ and let $k_j = 2^{Jn} j^{-1}$, $j = J+1, ..., L$. Then again $k_j \leqslant 2M_j \sim 2^{jn}$ and (3.21) provides

$$e_{k_j}(id_j) \leqslant c\langle j\rangle^{-\varkappa}\, 2^{-J(n/p)} j^{1/p} (j-J)^{1/p} \leqslant c'\, 2^{-J(n/p)} j^{-\varkappa + 2/p}.$$

Thus with $L \sim J2^J > J$,

$$\begin{aligned}\sum_{j=J+1}^{L} e^{\varrho}_{k_j}(id_j) &\leqslant c\, 2^{-J(n/p)\,\varrho} \sum_{j=J+1}^{L} j^{-(\varkappa - 2/p)\,\varrho} \\ &= c\, 2^{-J(n/p)\,\varrho} \sum_{j=J+1}^{L} j^{-1} \leqslant C 2^{-J(n/p)\,\varrho} J.\end{aligned}$$

On the other hand,

$$k \leqslant c\, 2^{Jn} + \sum_{j=J+1}^{L} k_j = c\, 2^{Jn} + 2^{Jn} \sum_{j=J+1}^{L} j^{-1} \leqslant c'\, 2^{Jn} \log \frac{L}{J} \leqslant CJ2^{Jn},$$

and so (3.22) implies

$$e_{cJ2^{Jn}}(id_{p,\infty}) \leqslant c' J^{1/\varrho} 2^{-J(n/p)} = c' (J 2^{Jn})^{-1/p}\, J^{1/\varrho + 1/p}.$$

This yields the second line in (3.16). Assume finally $0 < \varkappa < \frac{1}{\varrho} + \frac{2}{p}$. Choose $v < 1$ such that $\varkappa < \frac{1}{\varrho} + \frac{v+1}{p} < \frac{1}{\varrho} + \frac{2}{p}$ and put $k_j = 2^{Jn} j^{-v} < 2M_j \sim 2^{jn}$, $j = J+1, ..., L$. Thus we obtain

$$k \leqslant c\, 2^{Jn} + \sum_{j=J+1}^{L} k_j = c\, 2^{Jn} + 2^{Jn} \sum_{j=J+1}^{L} j^{-v} \leqslant c'\, 2^{Jn} L^{1-v}, \tag{3.24}$$

and

$$\begin{aligned}\sum_{j=J+1}^{L} e^{\varrho}_{k_j}(id_j) &\leqslant c\, 2^{-J(n/p)\,\varrho} \sum_{j=J+1}^{L} j^{-(\varkappa - (v+1)/p)\,\varrho} \\ &\leqslant c'\, 2^{-J(n/p)\,\varrho} L^{1-(\varkappa - (v+1)/p)\,\varrho}.\end{aligned} \tag{3.25}$$

Assume now that $2^{J(n/p)} \sim L^{1/\varrho + (v+1)/p}$; thus by (3.24), $k \leqslant cL^{p/\varrho + 2}$, and by (3.25)

$$\sum_{j=J+1}^{L} e^{\varrho}_{k_j}(id_j) \leqslant cL^{-\varkappa\varrho}.$$

Hence we may continue (3.22) by

$$e^{\varrho}_{cL^{2+p/\varrho}}(id_{p,\infty}) \leqslant c'(L^{-(1/\varrho+(\nu+1)/p)\,\varrho}(\log L)^{-\varkappa\varrho}+L^{-\varkappa\varrho}) \leqslant CL^{-\varkappa\varrho}.$$

The last line in (3.16) is established. ▌

Remark 3.6. Obviously our results (3.13), (3.14), and (3.16) are not sharp, even in the Banach space setting ($p, q \geqslant 1$). One can certainly improve both estimates in, say, the Hilbert space setting ($p=q=2$), by application of some deep results about the ℓ-norm and related results for Kolmogorov and entropy numbers; we refer to the book of Pisier [29, Chap. 5] for an excellent presentation of all the necessary background material as well as details, and to the papers of Gluskin [17], Sudakov [31], and Pajor and Tomczak-Jaegermann [26, 27]. Moreover, Carl and Pajor proved in [6] some similar result when replacing the Gaussian variables by random choices of signs. So there are several possible ways in which Propositions 3.4 and 3.5 might be improved, at least when Hilbert or Banach spaces are involved. However, as we have not yet obtained final (that is, sharp) results by following these procedures (though minor improvements for some values of $\varkappa$ do result) and wish to keep this paper at a reasonable length, we shall not present these further investigations here. An extensive study of $e_k(id_{p,\infty})$, $0<p<\infty$, or, more general, $e_k(id_{p_1,p_2})$, $0<p_1<p_2\leqslant\infty$, also with different q-parameters in (3.2), is thus postponed to a later occasion.

We give the counterpart of Propositions 3.4 and 3.5 in terms of approximation numbers. Recall notation (3.12).

PROPOSITION 3.7. *Let $\varkappa>0$, $1<p\leqslant\infty$, $0<q\leqslant\infty$ and $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$. Then*

$$a_k(id_{p,\infty})\sim(\log\langle k\rangle)^{-\varkappa}, \qquad k\in\mathbb{N}. \tag{3.26}$$

Proof. The case $p=\infty$ is covered by (3.4), so assume $p<\infty$ now. The proof is a simple modification of the corresponding one for $p=\infty$, i.e., Proposition 3.3 above. We start with the upper estimate in (3.26). The multiplicativity of approximation numbers together with the monotonicity of ℓ_q-spaces imply

$$a_{k+1}(id_{p,\infty})\leqslant ca_k(id_{\infty,\infty}),$$

because $\ell_q(\ell_p^{M_j})\hookrightarrow\ell_q(\ell_\infty^{M_j})$, $0<p,\ q\leqslant\infty$. Thus the upper estimate in (3.26) follows by (3.4) with $p=\infty$. Conversely, dealing with the corresponding estimate from below, we closely follow Step 1 of the proof of

Proposition 3.3. We have the adapted diagram (3.15) instead of (3.8) again and conclude by the same arguments as above, that

$$a_k(id_{p,\infty}) \geqslant c\langle j\rangle^{-\varkappa}\, a_k(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}),$$

see (3.9). Note that

$$a_k(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j}) \geqslant c,$$

where $k \leqslant \frac{1}{4} M_j^{2/p'}$ for $1 < p \leqslant 2$ and $k \leqslant \frac{1}{4} M_j$ for $2 \leqslant p \leqslant \infty$, cf. [15, Corollary 3.2.3, p. 103; 4, Corollary 2.2(ii)]. Since $M_j \sim 2^{jn}$, hence $a_m(id_{p,\infty}) \geqslant c(\log\langle m\rangle)^{-\varkappa}$. ▮

Corollary 3.8. *Let $\varkappa > 0$, $0 < p \leqslant 1$, $0 < q \leqslant \infty$, and $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. There are positive numbers c_1 and c_2 such that for all $k \in \mathbb{N}$,*

$$c_1 k^{-1/2}(\log\langle k\rangle)^{-\varkappa} \leqslant a_k(id_{p,\infty}) \leqslant c_2(\log\langle k\rangle)^{-\varkappa}. \tag{3.27}$$

Proof. The upper estimate follows by monotonicity from (3.4) again. Concerning the estimate from below, we use

$$a_k(id\colon \ell_\infty^{M_j} \to \ell_2^{M_j}) = (M_j - k + 1)^{1/2}, \qquad k = 1, ..., M_j,$$

see [28, p. 109]. Moreover, $a_k(id\colon \ell_p^{M_j} \to \ell_2^{M_j}) \sim 1$, $1 \leqslant k \leqslant \frac{1}{4} M_j$, $0 < p < 2$, see [15, Theorem 3.2.3/2(i), p. 109]. The multiplicativity of approximation numbers yields

$$\begin{aligned} c_1 &\leqslant a_{M_j/4}(id\colon \ell_p^{M_j} \to \ell_2^{M_j}) \\ &\leqslant c_2 a_{M_j/8}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j})\, a_{M_j/8}(id\colon \ell_\infty^{M_j} \to \ell_2^{M_j}) \\ &\leqslant c_3 a_{M_j/8}(id\colon \ell_p^{M_j} \to \ell_\infty^{M_j})\, M_j^{1/2}. \end{aligned}$$

This covers the lower estimate in (3.27). ▮

Remark 3.9. It might be possible to replace (3.27) by (3.26) when $0 < p \leqslant 1$, too, by using some recently developed interpolation technique of Cobos and Signes for approximation numbers, see [9, Lemma 4.7(b)]. Their result as well as some more details about a possible application can be found in Remark 3.15 below.

Note that Leopold obtained in [23, Theorem 4] similar results when dealing with the more general setting

$$id_p^q\colon \ell_{q_1}(\ell_{p_1}^{M_j}) \to \ell_{q_2}(\langle j\rangle^{-\varkappa}\, \ell_{p_2}^{M_j}),$$

where $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < q_1, q_2 \leqslant \infty$, $\varkappa > (\frac{1}{q_2} - \frac{1}{q_1})_+$, and $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$.

3.2. *Embeddings in Lipschitz Spaces; the Limiting Case*

Recall our notation id^B for the embedding

$$id^B\colon B^{1+n/p}_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U), \tag{3.28}$$

where $0<p\leqslant\infty$, $0<q\leqslant\infty$, $\alpha>1/q'$, and U is the unit ball in $\mathbb{R}^n$. We deal with the "limiting" case in the sense that we study embeddings between spaces where the differential dimensions are the same, that is, $s_1-n/p_1=s_2-n/p_2$, where $s_1\geqslant s_2$ and $0<p_1\leqslant p_2\leqslant\infty$.

THEOREM 3.10 [13, Theorem 4.10]. *Let $0<q\leqslant\infty$ and $\alpha>1/q'$. Then there are positive numbers c_1 and c_2 such that for all $k\in\mathbb{N}$,*

$$c_1(\log\langle k\rangle)^{-\alpha}\leqslant e_k(id\colon B^1_{\infty,q}(U)\to \mathrm{Lip}^{(1,-\alpha)}(U))\leqslant c_2(\log\langle k\rangle)^{-\alpha+1/q'}.$$

In particular, when $0<q\leqslant 1$ (and thus $\alpha>0$), we obtain

$$e_k(id\colon B^1_{\infty,q}(U)\to \mathrm{Lip}^{(1,-\alpha)}(U))\sim(\log\langle k\rangle)^{-\alpha}.$$

Moreover, we have proved in [13, Theorem 4.10] that for $0<p\leqslant\infty$, $0<q\leqslant\infty$ and $\alpha>1/q'$ there are positive numbers c_1 and c_2 such that for all $k\in\mathbb{N}$,

$$c_1k^{-1/p}(\log\langle k\rangle)^{-\alpha}\leqslant e_k(id^B)\leqslant c_2(\log\langle k\rangle)^{-\alpha+1/q'}. \tag{3.29}$$

However, when $p<\infty$ we can replace (3.29) by a better estimate.

THEOREM 3.11. *Let $0<p<\infty$, $0<q\leqslant\infty$, $\alpha>1/q'$. Let $\varrho=\min(q,1)$. There are positive numbers c_1 and c_2 such that for all $k\in\mathbb{N}$,*

$$\begin{aligned} &c_1k^{-1/p}(\log\langle k\rangle)^{-\alpha}\\ &\quad\leqslant e_k(id^B)\\ &\quad\leqslant c_2\begin{cases} k^{-1/p}(\log\langle k\rangle)^{-\alpha+1/q'+1/\varrho+2/p}, & \alpha>\dfrac{1}{q'}+\dfrac{1}{\varrho}+\dfrac{2}{p}\\ k^{-1/p}(\log\langle k\rangle)^{1/\varrho+2/p}, & \alpha=\dfrac{1}{q'}+\dfrac{1}{\varrho}+\dfrac{2}{p}\\ k^{-(\alpha-1/q')/(2+p/\varrho)}, & \alpha<\dfrac{1}{q'}+\dfrac{1}{\varrho}+\dfrac{2}{p}. \end{cases}\end{aligned}$$

Proof. *Step* 1. The estimate from below is already covered by [13, Theorem 4.10], see (3.29). However, we want to give an alternative proof using characterisation (1.6) which was quite recently obtained by Krbec and Schmeisser. Moreover, we involve the non-limiting result concerning entropy numbers of the compact embedding

$$id_\Omega\colon B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega),$$

where $-\infty < s_2 < s_1 < \infty$, $0 < p_1, p_2 \leqslant \infty$, with $s_1 - s_2 > n(1/p_1 - 1/p_2)_+$, $0 < q_1, q_2 \leqslant \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain in $\mathbb{R}^n$. Here Edmunds and Triebel proved that

$$e_k(id_\Omega) \sim k^{-(s_1 - s_2)/n}, \qquad k \in \mathbb{N}, \tag{3.30}$$

see [15, Theorem 3.3.3/2, p. 118]. In particular, (3.30) implies that for all λ, $0 < \lambda < 1$,

$$e_k(id\colon B^{1+n/p}_{p,q}(U) \to \mathscr{C}^{1-\lambda}(U)) \sim k^{-1/p - \lambda/n}, \qquad k \in \mathbb{N}, \tag{3.31}$$

recall $\mathscr{C}^s = B^s_{\infty,\infty}$, $s > 0$, cf. [33, Theorem 2.5.7(ii), p. 90]. Thus there is some $c > 0$ such that for all $\lambda \in (0, 1)$ and all $k \in \mathbb{N}$,

$$e_k(id\colon B^{1+n/p}_{p,q}(U) \to \mathscr{C}^{1-\lambda}(U)) \geqslant c k^{-1/p - \lambda/n}. \tag{3.32}$$

The independence of $c > 0$ from $\lambda \in (0, 1)$ can be obtained by an interpolation argument, see Remark 3.12 below. Thus the multiplicativity of entropy numbers together with (1.6) yields

$$e_k(id^B) \geqslant c k^{-1/p - \lambda/n} \lambda^\alpha, \tag{3.33}$$

where λ with $0 < \lambda < 1$ may be suitably chosen. Note that

$$\max_{0 < \lambda < 1} k^{-\lambda/n} \lambda^\alpha = \left(\frac{\alpha n}{e}\right)^\alpha (\log k)^{-\alpha}, \tag{3.34}$$

where the maximum is taken for $\lambda_0 = \alpha n (\log k)^{-1}$, $k \in \mathbb{N}$, $k \geqslant 2$. Thus for (large enough) $k \in \mathbb{N}$ we have $\lambda_0 \in (0, 1)$ and (3.33) leads to

$$e_k(id^B) \geqslant c k^{-1/p} (\log \langle k \rangle)^{-\alpha}. \tag{3.35}$$

Step 2. It remains to show the estimates from above. We benefit from our improved assertions for entropy numbers in sequence spaces, that is, Proposition 3.5, whereas we had the special case Proposition 3.2 in [13] only. We face the problem now that we do not want to repeat the whole proof of [13, Theorem 3.5] (dealing with the upper estimates for the entropy numbers) in detail. Roughly speaking, the crucial trick is to find (non-linear) bounded operators S and T such that we obtain the following commutative diagram,

$$\begin{array}{ccc} B^{1+n/p}_{p,q}(U) & \xrightarrow{\quad S \quad} & \ell_q(\ell_p^{M_k}) \\ {\scriptstyle id^B}\big\downarrow & & \big\downarrow{\scriptstyle id_{p,\infty}} \\ \mathrm{Lip}^{(1,-\alpha)}(U) & \xleftarrow{\quad T \quad} & \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_\infty^{M_k}) \end{array} \tag{3.36}$$

This is done via atomic (or, strictly speaking, even quarkonial) decompositions of function spaces, but we do not propose to go into further details here because of the general similarity of the arguments to those given in [13]: see the proof of Theorem 3.5 in [13] and [35, Sects. 13, 14, 20] for more information. Note that as the operator S is non-linear we cannot use the same method of proof when studying approximation numbers later. The method of atomic decompositions of function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$ is studied in detail in [35, Sect. 13]; apart from definitions and basic properties we essentially rely on the characterisation [35, Theorem 13.8, p. 75]. There a mechanism is established by which distributions $f \in B^s_{p,q}(\mathbb{R}^n)$ can be transformed into a sequence of complex numbers belonging to some space $\ell_q(\ell_p^{M_k})$, simultaneously controlling the corresponding norms. This provides the boundedness of the operator S. Note that we used the same notation (for the corresponding operators), that is, S and T, as in [35, Proposition 20.5, pp. 162–165; 13, Theorem 3.5]. Thus diagram (3.36) has its counterpart in [13, (3.20), (3.26)]. Concerning the independence of the "inverse" operator T from the used atomic decomposition, one has to involve even "smaller" building blocks than atoms, i.e., "quarks"; cf. [35, Sect. 14] for all necessary details. Moreover, one also needs some "quarkonial version" of Propositions 3.2 and 3.5 then, but this can be obtained without difficulties; cf. [35, Sect. 9; 13, Corollary 3.3]. The whole procedure, including the description of the operator T, can be found in Steps 2 and 3 of the proof of Theorem 3.5 in [13]. More precisely, we dealt in [13] with the slightly modified version of (3.36) when T is regarded as a map between $\ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_p^{M_k})$ and $\mathrm{Lip}^{(1,-\alpha)}(U)$ (recall that we only had Proposition 3.2 at this point). However, following the respective proof (i.e., Step 4 of the proof of Theorem 3.5 in [13]) one easily verifies that $T\colon \ell_q(\langle k\rangle^{-(\alpha-1/q')}\ell_\infty^{M_k}) \to \mathrm{Lip}^{(1,-\alpha)}(U)$ remains bounded. Thus by the multiplicativity of entropy numbers and $id^B = T \circ id_{p,\infty} \circ S$, Proposition 3.5 concludes the proof. ∎

Remark 3.12. We want to explain why the constant $c > 0$ in (3.32) is not only independent of $k \in \mathbb{N}$, but does not depend upon $\lambda \in (0, 1)$ either. This is an easy consequence of some interpolation argument in the sense of [15, Theorem 1.3.2(i), p. 13]: One can obtain $e_k(id\colon B^{1+n/p}_{p,q}(U) \to B^0_{\infty,\infty}(U))$ via interpolation from $e_k(id\colon B^{1+n/p}_{p,q}(U) \to \mathscr{C}^{1-\lambda}(U))$ and $e_k(id\colon B^{1+n/p}_{p,q}(U) \to B^{-1}_{\infty,\infty}(U))$ where all appearing constants may depend upon p, q and n, but not on λ. This yields the independence of the constants in (3.32) and (3.33) of $\lambda \in (0, 1)$. Following the maximisation procedure we finally get that c in (3.35) does not depend upon λ_0, i.e. upon $k \in \mathbb{N}$.

As we pointed out after Proposition 1.5 one has to deal carefully with characterisation (1.6) because the spaces $\mathscr{C}^{1-\lambda}$ involved there are defined via first differences which can cause trouble when $\lambda \downarrow 0$. However, by similar interpolation arguments (for the non-limiting setting) as above and the fact that $B^{1-\lambda}_{\infty,1} \hookrightarrow \mathscr{C}^{1-\lambda} \hookrightarrow B^{1-\lambda}_{\infty,\infty}$ when $0 < \lambda < 1$ (i.e., the embedding constants do not depend upon $\lambda > 0$), one can safely surmount this trap.

We give the counterpart of Theorems 3.10, 3.11 in terms of approximation numbers.

THEOREM 3.13. *Let* $1 < p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha > 1/q'$. *Then there are positive numbers* c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1(\log \langle k \rangle)^{-\alpha} \leqslant a_k(id^B) \leqslant c_2(\log \langle k \rangle)^{-\alpha + 1/q'}.$$

In particular, when $0 < q \leqslant 1$, *we obtain*

$$a_k(id^B) \sim (\log \langle k \rangle)^{-\alpha}.$$

Proof. The estimate from below can be obtained in the same way as in Step 1 in the proof of Theorem 3.11, where (3.32) has now to be replaced by

$$a_k(id\colon B^{1+n/p}_{p,q}(U) \to \mathscr{C}^{1-\lambda}(U)) \geqslant c \begin{cases} k^{-\lambda/n}, & p \geqslant 2 \\ k^{-(\lambda/n)(p'/2)}, & 1 < p \leqslant 2, \lambda \leqslant n/p' \\ k^{-(\lambda/n)-(1/2)}, & 0 < p \leqslant 1, \end{cases} \tag{3.37}$$

see [15, Theorem 3.3.4, p. 119; 4, Theorem 3.1]. Note that the above constant $c > 0$ is independent of $\lambda \in (0, 1)$, as we may use again some interpolation argument parallel to Remark 3.12. This is based on some very recent result of Cobos and Signes [9, Lemma 4.7] dealing with interpolation assertions for approximation numbers in the sense of [15, Theorem 1.3.2(i), p. 13] (which is related to entropy numbers), see Remark 3.15 below. We deal with the upper estimates now. Recall our proof of Proposition 4.7 in [13], especially Steps 2 and 3, concerning estimates for the entropy numbers of $B^s_{p,q_1}(\Omega) \hookrightarrow B^{(s,-b)}_{p,q_2}(\Omega)$, where $b > (1/q_2 - 1/q_1)_+$. (Note, that we have modified there the construction given by Leopold in [22] when he studied the similar situation $id\colon B^{(s,b)}_{p,q_1}(\Omega) \hookrightarrow B^s_{p,q_2}(\Omega)$.) Assume $\Omega \subset \{x \in \mathbb{R}^n : |x_j| \leqslant 1,\ j = 1, ..., n\}$ (which is trivial in our model case $\Omega = U$). Roughly speaking, one constructs some decomposition of a function $f \in B^s_{p,q_1}(\mathbb{R}^n)$, $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x_j| \leqslant 1,\ j = 1, ..., n\}$, $\|f \mid B^s_{p,q_1}(\mathbb{R}^n)\| \leqslant 1$, into $f = f^N + f_{N,1} + f_{N,2}$, where the rank of the linear operator $f \mapsto f_{N,1}$ can be estimated from above by $c2^{nN}$. Moreover, we obtain by [13, (4.25), (4.26)] that $\|f - f_{N,1} \mid B^{(s,-b)}_{p,q_2}(\mathbb{R}^n)\| \leqslant c\langle N\rangle^{-b+(1/q_2-1/q_1)_+}$. In other words, there is some $c > 0$ such that for all $k \in \mathbb{N}$,

$$a_k(id\colon B^s_{p,q_1}(\Omega) \to B^{(s,-b)}_{p,q_2}(\Omega)) \leqslant c(\log\langle k\rangle)^{-b+(1/q_2-1/q_1)_+}.$$

The related assertion concerning the above-described situation studied by Leopold can be found in [22, Remark 4]. Recall $B^{1+n/p}_{p,q} \hookrightarrow B^1_{\infty,q}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Thus by the multiplicativity of approximation numbers and (2.4),

$$\begin{aligned} a_k(id^B) &\leqslant c a_k(id\colon B^1_{\infty,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \\ &\leqslant c' a_k(id\colon B^1_{\infty,q}(U) \to B^{(1,-\alpha)}_{\infty,1}(U)) \\ &\leqslant C\,(\log\langle k\rangle)^{-\alpha+1/q'} \end{aligned}$$

for some $C > 0$ and all $k \in \mathbb{N}$. ∎

Note that in case of approximation numbers we benefit far less from our preceding results in Subsection 3.2. This is due to the (partly) non-linear procedure in the proof of Theorem 3.11, that is, the operator S in (3.36), which cannot be used for approximation numbers; recall also Step 2 of the proof of Theorem 3.11.

Remark 3.14. In view of (3.37) we obtain for $0 < p \leqslant 1$, $0 < q \leqslant \infty$, $\alpha > 1/q'$, that there are positive numbers c_1 and c_2 such that for all $k \in \mathbb{N}$,

$$c_1 k^{-1/2}(\log\langle k\rangle)^{-\alpha} \leqslant a_k(id^B) \leqslant c_2(\log\langle k\rangle)^{-\alpha+1/q'}.$$

Remark 3.15. We return to the result of Cobos and Signes in [9] mentioned above. In particular, they have proved in [9, Lemma 4.7(a)] the following: *Let A be a Banach space, let $\bar{B} = (B_0, B_1)$ be a quasi-linearisable couple, let B be an intermediate space with respect to $\bar{B}$ and let* $\varrho(t) = \inf\{J(t,b)\colon b \in B_0 \cap B_1,\ \|b\|_B = 1\}$, *where $J(t,\cdot)$ is the Peetre J-functional,* *i.e.,* $J(t, b) = \max\{\|b\|_{B_0}, t\,\|b\|_{B_1}\}$ *where the maximum is taken over all* $b \in B_0 \cap B_1$. *If* $T \in \mathscr{L}(A, B_0 \cap B_1)$ *and* $n_0, n_1 \in \mathbb{N}$, *then*

$$a_{n_0+n_1-1}(T_{A,B}) \leqslant c\,a_{n_0}(T_{A,B_0})\,\varrho^*\left(\frac{a_{n_1}(T_{A,B_1})}{a_{n_0}(T_{A,B_0})}\right),$$

for some constant $c > 0$ *and with* $\varrho^*(t) = 1/\varrho(t^{-1})$.

The notion of quasi-linearisable couples was introduced by Peetre; for definitions see [32, Sect. 1.8.4, pp. 51–53; 9, Sect. 2]. For our purpose it is sufficient to note that $(B^{s_0}_{p,q}(\mathbb{R}^n), B^{s_1}_{p,q}(\mathbb{R}^n))$ satisfies this condition. This can be derived from related results about positive operators and quasi-linearisable interpolation couples, see [32, Theorem 1.13.2, p. 77, Theorem 1.14.2, p. 92]. The essential point is that in case of the same p and q in both spaces (in our situation $p = q = \infty$) there is a lift operator $I_s f = \mathscr{F}^{-1}\langle\xi\rangle^s \mathscr{F}f$, $f \in S'(\mathbb{R}^n)$, $s \in \mathbb{R}$, mapping $B^{\sigma}_{p,q}(\mathbb{R}^n)$ isomorphically onto $B^{\sigma-s}_{p,q}(\mathbb{R}^n)$, $\sigma \in \mathbb{R}$, see [33, Theorem 2.3.8]. Now in connection with [32, Sect. 2.5] one verifies that indeed $(B^{s_0}_{p,q}(\mathbb{R}^n), B^{s_1}_{p,q}(\mathbb{R}^n))$ is a quasi-linearisable interpolation couple. Moreover, Cobos and Signes have also established a similar assertion related to the situation when the domain space is "interpolated," cf. [9, Lemma 4.7(b)].

Remark 3.16. We briefly want to compare our *limiting* results, i.e., Theorems 3.10, 3.11, and 3.13, with their non-limiting counterparts. One possibility to "approximate" our limiting embedding id^B by non-limiting embeddings of a similar type is shown in the $(s, \frac{1}{p})$-diagram of Fig. 1. Any space $B^s_{p,q}$ or $F^s_{p,q}$ is characterised there by its pair of parameters $(s, \frac{1}{p})$ (independent of q, $0 < q \leqslant \infty$), as usual. In that (rough) sense our target space $\mathrm{Lip}^{(1,-\alpha)}(U)$ can be found at the point $(1, 0)$, too (neglecting the additional smoothness provided by the log-exponent $\alpha \geqslant 0$).

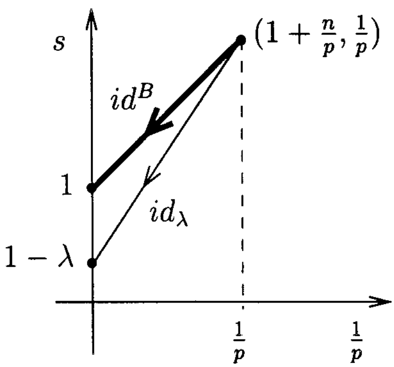

FIGURE 1

In our situation described above we stick at the parameter $p_2=\infty$ for the target space, but have less smoothness, say, $s_2=1-\lambda<1$, $\lambda>0$. Thus we are interested in assertions about $e_k(id_\lambda)$ and $a_k(id_\lambda)$ when $\lambda\downarrow 0$ and id_λ is given by

$$id_\lambda\colon B^{1+n/p}_{p,q}(U)\to B^{1-\lambda}_{\infty,\infty}(U),$$

where $0<p\leqslant\infty$, $0<q\leqslant\infty$, and $\lambda>0$. Note that one has for any $k\in\mathbb{N}$ and $\lambda>0$,

$$e_k(id_\lambda)\sim k^{-1/p-\lambda/n},$$
$$a_k(id_\lambda)\sim\begin{cases}k^{-\lambda/n}, & p\geqslant 2\\ k^{-(\lambda/n)(p'/2)}, & 1<p\leqslant 2,\quad \lambda<\dfrac{n}{p'},\end{cases} \tag{3.38}$$

where $1/p+1/p'=1$, as usual; cf. [15, Theorem 2, p. 118] concerning entropy numbers, and [15, Theorem 3.3.4, p. 119; 4] for the approximation numbers. (Note that we are only interested in small numbers $\lambda>0$.) In view of (3.38) (for $\lambda\downarrow 0$) it is thus rather natural that in case of entropy numbers the extra term $k^{-1/p}$ appears in the limiting situation, too, in contrast to the case of approximation numbers; see Theorems 3.11 and 3.13.

Now we study the (asymptotic behaviour of the) entropy numbers of the compact embedding $id_{\alpha\beta}$, $\beta>\alpha>0$; recall Proposition 2.6.

THEOREM 3.17. *Let $\beta>\alpha>0$. Then*

$$e_k(id_{\alpha\beta}\colon \mathrm{Lip}^{(1,-\alpha)}(U)\to\mathrm{Lip}^{(1,-\beta)}(U))\sim(\log\langle k\rangle)^{-(\beta-\alpha)},\qquad k\in\mathbb{N}. \tag{3.39}$$

Proof. Note that (3.39) implies Proposition 2.6 again by the properties of entropy numbers.

Step 1. We prove the estimate from below in (3.39). Equation (2.4) implies for $\beta\geqslant\alpha>0$ that

$$B^{(1,-\alpha)}_{\infty,1}(U)\hookrightarrow\mathrm{Lip}^{(1,-\alpha)}(U)\hookrightarrow\mathrm{Lip}^{(1,-\beta)}(U)\hookrightarrow B^{(1,-\beta)}_{\infty,\infty}(U).$$

We use Leopold's result [22, Theorem 2] together with the multiplicativity of entropy numbers and obtain

$$\begin{aligned}e_k(id_{\alpha\beta})&\geqslant c\,e_k(id\colon B^{(1,-\alpha)}_{\infty,1}(U)\to B^{(1,-\beta)}_{\infty,\infty}(U))\\&\geqslant c'(\log\langle k\rangle)^{-(\beta-\alpha)},\qquad k\in\mathbb{N}.\end{aligned}$$

Step 2. The estimate from above is more difficult to handle. We first show that

$$e_k(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu^\beta \mathscr{C}^{1-\mu}(U)) \leqslant c(\log \langle k \rangle)^{-(\beta-\alpha)}$$

if $\beta > \alpha > 0$ and $\mu > \mu_0 = \alpha n(\log k)^{-1}$, $k \geqslant k_0$ large, where $c > 0$ depends upon α, β only. We conclude from (1.6) and (3.31) (with $p = q = \infty$) that

$$\begin{aligned} &e_k(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu^\beta \mathscr{C}^{1-\mu}(U)) \\ &\quad\leqslant c\lambda^{-\alpha} e_k(id\colon \mathscr{C}^{1-\lambda}(U) \to \mathscr{C}^{1-\mu}(U))\, \mu^\beta \\ &\quad\leqslant c'\lambda^{-\alpha} \mu^\beta k^{-(\mu-\lambda)/n} \\ &\quad= c'\lambda^{-\alpha} k^{\lambda/n} \mu^\beta k^{-\mu/n}, \end{aligned} \tag{3.40}$$

where $\mu > \lambda > 0$. Straightforward calculation shows that

$$\min_{0<\lambda<1} k^{\lambda/n} \lambda^{-\alpha} = \left(\frac{e}{\alpha n}\right)^\alpha (\log k)^\alpha, \tag{3.41}$$

where the minimum is taken at $\lambda_0 = \alpha n(\log k)^{-1} < 1$ when $k \geqslant k_0$. Thus for large $k \in \mathbb{N}$, we may continue (3.40) by

$$e_k(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu^\beta \mathscr{C}^{1-\mu}(U)) \leqslant c(\log \langle k \rangle)^\alpha \mu^\beta k^{-\mu/n}, \qquad \mu > \lambda_0.$$

Moreover, $\mu^\beta k^{-\mu/n}$ is bounded from above by $(\frac{\beta n}{e})^\beta (\log k)^{-\beta}$ for all $\mu > 0$; see (3.34). We consequently arrive at

$$e_k(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu^\beta \mathscr{C}^{1-\mu}(U)) \leqslant c_{\alpha\beta}(\log \langle k \rangle)^{-(\beta-\alpha)},$$

$$\mu > \mu_0 = \frac{\alpha n}{\log k}. \tag{3.42}$$

Step 3. We finish the proof of the upper estimate in (3.39). Let $\mu_j = 1/j$, and $k_j \sim 2^{J\alpha n}$, $j = 1, ..., J$, where $J \in \mathbb{N}$. Thus $\mu_j > \alpha n(\log k_j)^{-1}$ for $1 \leqslant j \leqslant J$, and (3.42) yields

$$e_{k_j}(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu_j^\beta \mathscr{C}^{1-\mu_j}(U)) \leqslant c\langle J \rangle^{-\beta-\alpha}, \qquad j = 1, ..., J.$$

We follow the idea of the proof of Theorem 3.4.2 in [15], especially Step 5; see [15, pp. 136–137], where Edmunds and Triebel studied a similar limiting situation. For convenience we write $e_{k_j} = e_{k_j}(id\colon \mathrm{Lip}^{(1,\,-\alpha)}(U) \to \mu_j^\beta \mathscr{C}^{1-\mu_j}(U))$ for the moment. Recall notation (2.15). Cover the unit ball U_α of $\mathrm{Lip}^{(1,\,-\alpha)}(U)$ by 2^{k_1} balls in $\mu_1^\beta \mathscr{C}^{1-\mu_1}(U)$ of radius $2e_{k_1}$, each ball having centre in U_α. Let U_1 be one of these balls, and cover $U_\alpha \cap U_1$ in $\mu_2^\beta \mathscr{C}^{1-\mu_2}(U)$ by 2^{k_2} balls of radius $2e_{k_2}$, where we may assume that the centres are in $U_\alpha \cap U_1$. By iteration, we obtain a covering of $U_\alpha \cap U_1 \cap \cdots \cap U_{J-1}$ in $\mu_J^\beta \mathscr{C}^{1-\mu_J}(U)$ by 2^{k_J} balls of radius $2e_{k_J}$, where the centres of these balls are in $U_\alpha \cap U_1 \cap \cdots \cap U_{J-1}$. Denoting these centres by g_ℓ, $\ell = 1, ..., L$, we obtain

$$L = 2^{k_1 + \cdots + k_J}, \qquad \text{where} \quad \sum_{j=1}^{J} k_j \sim J 2^{J\alpha n}. \tag{3.43}$$

Thus for given $f \in U_\alpha$ there is one of these centres g_ℓ such that

$$\mu_j^\beta \, \| f - g_\ell \mid \mathscr{C}^{1-\mu_j}(U) \| \leqslant c e_{k_j} \leqslant c' \langle J \rangle^{-(\beta-\alpha)}, \qquad j = 1, ..., J. \tag{3.44}$$

Note that by (1.6),

$$\| g \mid \mathscr{C}^{1-s}(U) \| = s^{-\alpha} s^{\alpha} \, \| g \mid \mathscr{C}^{1-s}(U) \| \leqslant c s^{-\alpha} \, \| g \mid \mathrm{Lip}^{(1,-\alpha)}(U) \|$$

for any $s \in (0, 1)$ and $g \in \mathrm{Lip}^{(1,-\alpha)}(U)$; we have

$$\| id \colon \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathscr{C}^{1-s}(U) \| \leqslant c s^{-\alpha}, \qquad s \in (0, 1), \tag{3.45}$$

where the constant c is independent of s. Let $j > J$, $\mu_j = 1/j$; thus the counterpart of (3.44) reads as

$$\begin{aligned} \mu_j^\beta \, \| f - g_\ell \mid \mathscr{C}^{1-\mu_j}(U) \| &\leqslant c \mu_j^{\beta-\alpha} = c j^{-(\beta-\alpha)} \\ &\leqslant c' \langle J \rangle^{-(\beta-\alpha)}, \qquad j > J, \end{aligned} \tag{3.46}$$

where we used (3.45) and $f, g_\ell \in \mathrm{Lip}^{(1,-\alpha)}(U)$. Let $k = \sum_{j=1}^J k_j \sim J 2^{J\alpha n}$; then (3.43), (3.44), and (3.46) give $e_{cJ2^{J\alpha n}}(id_{\alpha\beta}) \leqslant C \langle J \rangle^{-(\beta-\alpha)}$. Put $m \sim J 2^{J\alpha n} \in \mathbb{N}$; then $\log\langle m \rangle \sim J$ and we finally obtain the upper estimate in (3.39). This ends the proof. ∎

We return to the setting studied in Proposition 2.4.

COROLLARY 3.18. *Let* $\gamma - 1 > \beta > \alpha > 0$. *Then*

$$e_k(id \colon \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathscr{C}^{(1,-\beta)}(U)) \sim (\log\langle k \rangle)^{-(\beta-\alpha)}, \qquad k \in \mathbb{N}. \tag{3.47}$$

Furthermore, there are positive numbers c_1, c_2 *such that for all* $k \in \mathbb{N}$,

$$\begin{aligned} c_1 (\log\langle k \rangle)^{-(\gamma-\beta)} &\leqslant e_k(id \colon \mathscr{C}^{(1,-\beta)}(U) \to \mathrm{Lip}^{(1,-\gamma)}(U)) \\ &\leqslant c_2 (\log\langle k \rangle)^{-(\gamma-\beta)+1}. \end{aligned}$$

Proof. Concerning the upper estimate in (3.47) we have by Proposition 2.4 and Theorem 3.17 that

$$e_k(id\colon \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathscr{C}^{(1,-\beta)}(U)) \leqslant c e_k(\mathrm{Lip}^{(1,-\alpha)}(U) \to \mathrm{Lip}^{(1,-\beta)}(U)) \leqslant c'(\log\langle k\rangle)^{-(\beta-\alpha)}.$$

Conversely, using (2.7) and Leopold's result [13, Theorem 2] we estimate

$$c(\log\langle k\rangle)^{-(\beta-\alpha)} \leqslant e_k(id\colon B^{(1,-\alpha)}_{\infty,1}(U) \to B^{(1,-\beta)}_{\infty,\infty}(U)) \leqslant c' e_k(id\colon \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathscr{C}^{(1,-\beta)}(U))$$

and thus (3.47) is proved. Likewise we get

$$e_k(id\colon \mathscr{C}^{(1,-\beta)}(U) \to \mathrm{Lip}^{(1,-\gamma)}(U)) \leqslant c e_k(id\colon B^{(1,-\beta)}_{\infty,\infty}(U) \to B^{(1,-\gamma)}_{\infty,1}(U)) \leqslant c'(\log\langle k\rangle)^{-(\gamma-\beta)+1}.$$

Finally, Proposition 2.4 and Theorem 3.17 imply

$$c(\log\langle k\rangle)^{-(\gamma-\beta)} \leqslant e_k(id\colon \mathrm{Lip}^{(1,-\beta)}(U) \to \mathrm{Lip}^{(1,-\gamma)}(U)) \leqslant c' e_k(id\colon \mathscr{C}^{(1,-\beta)}(U) \to \mathrm{Lip}^{(1,-\gamma)}(U)).$$

This ends the proof. ∎

Note that Leopold obtained in [22, Theorem 2] similar estimates for the corresponding entropy numbers when he studied compact embeddings of logarithmic Besov spaces on bounded domains, see also [13, Proposition 4.7].

3.3. *Embeddings in Lipschitz Spaces; the Non-limiting Case*

Let all spaces be defined on $U = \{x \in \mathbb{R}^n\colon |x| < 1\} \subset \mathbb{R}^n$ in the sequel. For convenience we shall write $e_k(A \hookrightarrow B)$ instead of $e_k(id\colon A \to B)$, likewise for approximation numbers. We briefly collect what is known so far. The notion of "non-limiting" case simply means, that we now handle embeddings between function spaces where the difference of the corresponding differential dimensions is strictly positive, i.e., $s_1 - n/p_1 / s_2 - n/p_2$.

Corollary 3.19. (i) *Let* $\alpha \geqslant 0$, $s > 0$, $0 < q \leqslant \infty$. *Then for all* $k \in \mathbb{N}$,

$$e_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow B^{1-s}_{\infty,q}) \sim a_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow B^{1-s}_{\infty,q}) \sim k^{-s/n}(\log\langle k\rangle)^{\alpha}. \tag{3.48}$$

In particular, we obtain for $s < 1$ *and all* $k \in \mathbb{N}$,

$$e_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow \mathscr{C}^{1-s}) \sim a_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow \mathscr{C}^{1-s}) \sim k^{-s/n}(\log\langle k\rangle)^{\alpha}, \tag{3.49}$$

and

$$e_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow C) \sim a_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow C) \sim k^{-1/n}(\log\langle k\rangle)^{\alpha}. \tag{3.50}$$

(ii) *Let* $\alpha \geqslant 0$, $0<p\leqslant\infty$, $0<q\leqslant\infty$ *and* $s>1+n/p$. *Then for all* $k\in\mathbb{N}$,

$$e_k(B^s_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim k^{-(s-1)/n}(\log\langle k\rangle)^{-\alpha}, \tag{3.51}$$

and

$$a_k(B^s_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim (\log\langle k\rangle)^{-\alpha} \begin{cases} k^{-(s-1)/n+1/p}, & 2\leqslant p\leqslant\infty \\ k^{-(s-1)/n+1/2}, & 1<p<2, \quad s>n+1 \\ k^{-((s-1)/n-1/p)\,p'/2}, & 1<p<2, \quad s<n+1, \end{cases} \tag{3.52}$$

where $B^s_{p,q}$ *in* (3.51) *and* (3.52) *may be replaced by* $F^s_{p,q}$ (*when* $p<\infty$). *In particular*,

$$e_k(\mathscr{C}^s \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim a_k(\mathscr{C}^s \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim k^{-(s-1)/n}(\log\langle k\rangle)^{-\alpha}, \tag{3.53}$$

and for $m\in\mathbb{N}$, $m\geqslant 2$,

$$e_k(C^m \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim a_k(C^m \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim k^{-(m-1)/n}(\log\langle k\rangle)^{-\alpha}. \tag{3.54}$$

Finally, (3.51) *implies for* $1<p<\infty$, $s>1+n/p$ *and* $\alpha\geqslant 0$, *that for all* $k\in\mathbb{N}$,

$$e_k(H^s_p \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim k^{-(s-1)/n}(\log\langle k\rangle)^{-\alpha}, \tag{3.55}$$

whereas (3.52) *leads to*

$$a_k(H^s_p \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \sim (\log\langle k\rangle)^{-\alpha} \begin{cases} k^{-(s-1)/n+1/p}, & p\geqslant 2 \\ k^{-(s-1)/n+1/2}, & p<2, \quad s>n+1 \\ k^{-((s-1)/n-1/p)\,p'/2}, & p<2, \quad s<n+1. \end{cases} \tag{3.56}$$

Proof. Note that (3.49) and (3.53) follow from (3.48) and (3.51), (3.52), respectively, by the identity $\mathscr{C}^\sigma = B^\sigma_{\infty,\infty}$, $\sigma>0$, see [33, Theorem 2.5.7(ii), p. 90], whereas (3.50) and (3.54) result from the embeddings $B^m_{\infty,1} \hookrightarrow C^m \hookrightarrow B^m_{\infty,\infty}$, $m\in\mathbb{N}_0$, see [33, (2.5.7/2), (2.5.7/11), pp. 89–90]. Likewise (3.55) is a consequence of (3.51) (for $F^s_{p,q}$) and $F^s_{p,2}=H^s_p$, $s\in\mathbb{R}$, $1<p<\infty$, cf. [33, Theorem 2.56(i), p. 88]. Moreover, by the elementary embeddings $B^s_{p,u} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,v}$ if, and only if, $u \leqslant \min(p, q)$, $v \geqslant \max(p, q)$, it is sufficient to prove (3.48) and (3.51) for the B-case. We shall only deal with estimates for entropy numbers, but the case of approximation numbers can be handled completely analogously.

We begin with the upper estimate in (3.48). Using the characterisation (1.6) by Krbec and Schmeisser we obtain

$$
\begin{aligned}
e_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow B^{1-s}_{\infty,q}) &\leqslant c\lambda^{-\alpha} e_k(\mathscr{C}^{1-\lambda} \hookrightarrow B^{1-s}_{\infty,q}) \\
&\leqslant c'\lambda^{-\alpha} k^{\lambda/n} k^{-s/n} \\
&\leqslant C k^{-s/n} (\log\langle k\rangle)^{\alpha}.
\end{aligned}
$$

The argument is parallel to that used in Step 1 of the proof of Theorem 3.11. (Note that one can obtain an upper bound for $e_k(\mathscr{C}^{1-\lambda} \hookrightarrow B^{1-s}_{\infty,q})$ via interpolation from $e_k(\mathscr{C}^{1} \hookrightarrow B^{1-s}_{\infty,q})$ and, say, $e_k(\mathscr{C}^{1-\sigma} \hookrightarrow B^{1-s}_{\infty,q})$, for all $\lambda \in (0, \sigma)$ with $\sigma := \min(1, s/2)$. Again, as none of the constants appearing depends upon λ, the constant C does not depend upon the minimum λ_0, that is, upon $k \in \mathbb{N}$. In view of the number σ we shall additionally assume $k \geqslant k_0$.) Conversely, we obtain by (3.30) that

$$
\begin{aligned}
ck^{-s/n} &\leqslant e_{2k}(B^1_{\infty,1} \hookrightarrow B^{1-s}_{\infty,q}) \\
&\leqslant c' e_k(B^1_{\infty,1} \hookrightarrow B^{(1,-\alpha)}_{\infty,1})\, e_k(B^{(1,-\alpha)}_{\infty,1} \hookrightarrow B^{1-s}_{\infty,q}) \\
&\leqslant C(\log\langle k\rangle)^{-\alpha}\, e_k(\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow B^{1-s}_{\infty,q}),
\end{aligned}
$$

where we used the multiplicativity of entropy numbers again. The last inequality is covered by [22, Theorem 2] and (2.4). This gives the lower estimate in (3.48). We now deal with (3.51). Note that by (3.30) and Theorem 3.10 we have

$$
\begin{aligned}
e_{2k}(B^s_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) &\leqslant c e_k(B^s_{p,q} \hookrightarrow B^1_{\infty,v})\, e_k(B^1_{\infty,v} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \\
&\leqslant c' k^{-(s-1)/n} (\log\langle k\rangle)^{-\alpha},
\end{aligned}
$$

where $s-1 > n/p$ and $0 < v \leqslant 1$. It remains to show the converse inequality. We use (3.30) and characterisation (1.6) and conclude

$$
ck^{-(s-1)/n - \lambda/n} \leqslant e_k(B^s_{p,q} \hookrightarrow \mathscr{C}^{1-\lambda}) \leqslant c' e_k(id\colon B^s_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)})\, \lambda^{-\alpha}
$$

so that $e_k(B^s_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}) \geqslant ck^{-(s-1)/n} k^{-\lambda/n} \lambda^{\alpha}$. Now (3.34) finishes the proof. ∎

4. APPLICATIONS

In this last section we give a few ideas about how to apply our results. In particular, we shall concentrate on some model cases only in order to present the method itself; i.e., we study the eigenvalue distribution of some (degenerate) pseudodifferential operator

$$B = b \cdot b^{\Omega}(\cdot, D),$$

acting in some space $H_p^s(\Omega)$, where b belongs to some (logarithmic) Lipschitz space and $b^{\Omega}(\cdot, D)$ is in some Hörmander class (suitably adapted to our domain). Furthermore, we establish upper bounds for the eigenvalues of an operator

$$B = b_2 A^{-1} b_1,$$

where b_1 belongs to some space L_r, b_2 to some (logarithmic) Lipschitz space, and A^{-1} is the inverse of some properly elliptic differential operator.

As a preparation we start with some extension of the concept of spaces $\mathrm{Lip}^{(1,-\alpha)}$ to the setting of L_p, $p < \infty$. This will be needed first to obtain an assertion of the type of Hölder's inequality (suitably adapted to our situation); afterwards we shall make use of the spaces $\mathrm{Lip}_p^{(1,-\alpha)}$, $p < \infty$, which we introduced as well as the achieved results when dealing with the eigenvalue distribution of the model operators described above.

4.1. *Lipschitz Spaces with Metric $p < \infty$*

We want to extend Definition 1.1 slightly. Recall our notation for the difference operator Δ_h^m, $m \in \mathbb{N}_0$, $h \in \mathbb{R}^n$, in (1.2) as well as (1.3).

Definition 4.1. Let $\alpha \geqslant 0$, $1 \leqslant p \leqslant \infty$. The space $\mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)$ is defined as the set of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f \mid \mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)\| := \|f \mid L_p(\mathbb{R}^n)\| + \sup_{0<|h|<1/2} \frac{\|\Delta_h f \mid L_p(\mathbb{R}^n)\|}{|h| \, |\log |h||^{\alpha}} \tag{4.1}$$

is finite.

These spaces were introduced by DeVore and Lorentz in [10, Chap. 2, Sect. 9, p. 51] as $Lip(1, L_p)$ when $\alpha = 0$, $\mathbb{R}^n$ being replaced by some interval $[a, b] \subset \mathbb{R}$ and $0 < p \leqslant \infty$. Note that in the above notation $\mathrm{Lip}^{(1,-\alpha)} = \mathrm{Lip}_\infty^{(1,-\alpha)}$; but as long as there is no danger of confusion we shall continue to write $\mathrm{Lip}^{(1,-\alpha)}$ then (when $p = \infty$).

Proposition 4.2. *Let* $1 \leqslant p \leqslant \infty$, $\alpha > 0$.

(i) *Then* $f \in \mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)$ *if, and only if,* f *belongs to* $L_p(\mathbb{R}^n)$ *and there is some* $c > 0$ *such that for all* λ, $0 < \lambda < 1$,

$$\| f \mid B^{1-\lambda}_{p,\infty}(\mathbb{R}^n) \| \leqslant c\lambda^{-\alpha}.$$

Moreover,

$$\sup_{0<\lambda<1} \lambda^{\alpha} \, \| f \mid B^{1-\lambda}_{p,\infty}(\mathbb{R}^n) \| \tag{4.2}$$

is an equivalent norm in $\mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)$.

(ii) *Let* $0 < q \leqslant \infty$. *Then*

$$B^1_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n) \qquad \text{if} \quad \alpha \geqslant \frac{1}{q'}. \tag{4.3}$$

Proof. Note that when $p = \infty$, part (i) coincides with the result of Krbec and Schmeisser in [19, Proposition 2.5] (see also Proposition 1.5). Likewise one regains (a weaker version of) our embedding theorem [13, Theorem 2.1 (ii)] from (ii) when $p = \infty$, recall Theorem 2.1 (i). It remains to handle the case $1 \leqslant p < \infty$. Concerning (i) one simply has to adapt the proof of Krbec and Schmeisser in [19, Proposition 2.5] in a suitable manner. One can easily check that there is no difficulty at all. Thus we do not repeat it here. Note that we have the characterisation of $B^{1-\lambda}_{p,\infty}$, $0 < \lambda < 1$, by first differences again, which requires some care when using this characterisation later on (when $\lambda \downarrow 0$). We prove (ii). Unlike our proof of (4.3) when $p = \infty$ (in [13]) we do not use atomic decompositions of function spaces this time (though a modified proof might work here as well). We follow a similar argument to [13, Remark 2.4] using Marchaud's inequality as well as equivalent characterisations of $\mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$, via the modulus of continuity. Recall the definition of the difference operator Δ^m_h, $m \in \mathbb{N}_0$, $h \in \mathbb{R}^n$, in (1.2). Then the *r th modulus of continuity* (*or smoothness*) of a function $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, is defined by

$$\omega_r(f, t)_p = \sup_{|h| \leqslant t} \| \Delta^r_h f \mid L_p(\mathbb{R}^n) \|, \qquad t > 0,$$

see [2, Chap. 5, Definition 4.2, p. 332; 10, Chap. 2, Sect. 7, pp. 44–46]. Recall that

$$\| f \mid B^1_{p,q}(\mathbb{R}^n) \| \sim \| f \mid L_p(\mathbb{R}^n) \| + \left(\int_0^\infty \left[\frac{\omega_2(f,t)_p}{t} \right]^q \frac{dt}{t} \right)^{1/q} \tag{4.4}$$

(with the usual modification if $q = \infty$), see [2, Chap. 5, Definition 4.3, p. 332; 10, Chap. 2, Sect. 10, pp. 54–56] (where the Besov spaces are defined in that way) and [33, Theorem 2.5.12, p. 110] for what concerns the equivalence of Definition 1.3(i) (with $b=0$) and characterisation (4.4). By (4.1) we have

$$\|f \mid \mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)\| \sim \|f \mid L_p(\mathbb{R}^n)\| + \sup_{0<t<1/2} \frac{\omega_1(f,t)_p}{t\,|\log t|^{\alpha}}. \tag{4.5}$$

Now Marchaud's inequality states the following: let $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, $t>0$, and $k \in \mathbb{N}$; then

$$\omega_k(f,t)_p \leqslant \frac{k}{\log 2}\, t^k \int_t^{\infty} \frac{\omega_{k+1}(f,u)_p}{u^k}\frac{du}{u}, \tag{4.6}$$

see [2, Chap. 5, (4.11), p. 334; 10, Chap. 2, Theorem 8.1, p. 47] (the latter dealing with the one-dimensional case). In particular, assuming $k=1$, then (4.6) implies that there is some $c>0$ such that

$$\omega_1(f,t)_p \leqslant ct \int_t^{\infty} \frac{\omega_2(f,u)_p}{u}\frac{du}{u}, \tag{4.7}$$

for all $f \in L_p(\mathbb{R}^n)$ and $t>0$. Let all spaces be defined on $\mathbb{R}^n$ unless otherwise stated. Now (4.5) together with Marchaud's inequality (4.7) lead to

$$\begin{aligned}
\|f \mid \mathrm{Lip}_p^{(1,-\alpha)}\| &\leqslant c_1 \left\{ \|f \mid L_p\| + \sup_{0<t<1/2} \frac{1}{|\log t|^{\alpha}} \int_t^{\infty} \frac{\omega_2(f,u)_p}{u}\frac{du}{u} \right\} \\
&\leqslant c_2 \left\{ \|f \mid L_p\| + \sup_{0<t<1/2} \frac{1}{|\log t|^{\alpha}} \int_t^{1} \frac{\omega_2(f,u)_p}{u}\frac{du}{u} \right\} \\
&\leqslant c_3 \left\{ \|f \mid L_p\| + \sup_{0<t<1/2} \frac{1}{|\log t|^{\alpha}} \left(\int_t^{1} \left[\frac{\omega_2(f,u)_p}{u}\right]^q \frac{du}{u} \right)^{1/q} \right. \\
&\quad \left. \times \left(\int_t^1 \frac{du}{u} \right)^{1/q'} \right\} \\
&\leqslant c_4 \left\{ \|f \mid L_p\| + \left(\int_0^{\infty} \left[\frac{\omega_2(f,u)_p}{u}\right]^q \frac{du}{u} \right)^{1/q} \right. \\
&\quad \left. \times \sup_{0<t<1/2} \frac{1}{|\log t|^{\alpha}} \left(\int_t^1 \frac{du}{u} \right)^{1/q'} \right\},
\end{aligned}$$

where the second estimate comes from Hölder's inequality for $1 \leqslant q \leqslant \infty$ and $1 = 1/q + 1/q'$. Moreover,

$$\frac{1}{|\log t|^{\alpha}} \left(\int_t^1 \frac{du}{u} \right)^{1/q'} \xrightarrow{t \downarrow 0} 0 \qquad \text{when} \quad \alpha \geqslant 1/q',$$

and so the respective supremum over all small t, $0<t<1/2$, is bounded from above by some constant. Thus we finally arrive by (4.4) at

$$\| f \mid \mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)\| \leqslant C \, \| f \mid B^1_{p,q}(\mathbb{R}^n)\| \qquad \text{if} \quad \alpha \geqslant \frac{1}{q'},$$

which yields (4.3) for $1 \leqslant p, q \leqslant \infty$. The extension to $0<q\leqslant 1$ then simply comes from the monotonicity of B-spaces, that is, $B^s_{p,q_1} \hookrightarrow B^s_{p,q_2}$ if $s\in\mathbb{R}$, $0<p\leqslant\infty$, and $0<q_1\leqslant q_2\leqslant\infty$. ∎

For our later application we need some suitably adapted Hölder inequality in spaces $\mathrm{Lip}_p^{(1,-\alpha)}$.

PROPOSITION 4.3. *Let* $1\leqslant p, q\leqslant\infty$, *be such that* $0\leqslant \frac{1}{r}=\frac{1}{p}+\frac{1}{q}\leqslant 1$. *Let* α, $\beta\geqslant 0$. *Then*

$$\mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)\cdot \mathrm{Lip}_q^{(1,-\beta)}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}_r^{(1,-(\alpha+\beta))}(\mathbb{R}^n). \tag{4.8}$$

Proof. Let all spaces be defined on $\mathbb{R}^n$ unless otherwise stated. Assertion (4.8) has to be understood in the following sense. We have to show that there is some positive number c such that for all $f\in \mathrm{Lip}_p^{(1,-\alpha)}$ and all $g\in \mathrm{Lip}_q^{(1,-\beta)}$,

$$\| fg \mid \mathrm{Lip}_r^{(1,-(\alpha+\beta))}\| \leqslant c \, \| f\mid \mathrm{Lip}_p^{(1,-\alpha)}\| \; \| g \mid \mathrm{Lip}_q^{(1,-\beta)}\|. \tag{4.9}$$

But this follows from the classical Hölder inequality and Definition 4.1: Let $h\in\mathbb{R}^n$, $0<|h|<1/2$; then by the definition (1.2) of Δ_h and $r\geqslant 1$ we obtain

$$\|\Delta_h(fg)\mid L_r\| \leqslant \|(\Delta_h f)(\Delta_h g)\mid L_r\| + \|g(\Delta_h f)\mid L_r\| + \|f(\Delta_h g)\mid L_r\|.$$

We apply the classical Hölder inequality for $0\leqslant \frac{1}{r}=\frac{1}{p}+\frac{1}{q}\leqslant 1$ and obtain by our assumptions on f and g and Definition 4.1 that

$$\begin{aligned}
\|\Delta_h(fg)\mid L_r\| &\leqslant c(\|\Delta_h f\mid L_p\| \; \|\Delta_h g\mid L_q\| + \|g\mid L_q\| \; \|\Delta_h f\mid L_p\| \\
&\qquad + \|f\mid L_p\| \; \|\Delta_h g\mid L_q\|) \\
&\leqslant c(\|\Delta_h f\mid L_p\| \; \|\Delta_h g\mid L_p\| \\
&\qquad + \|g\mid \mathrm{Lip}_q^{(1,-\beta)}\| \; \|\Delta_h f\mid L_p\| + \|f\mid \mathrm{Lip}_p^{(1,-\alpha)}\| \; \|\Delta_h g\mid L_q\|) \\
&\leqslant c' \, \|f\mid \mathrm{Lip}_p^{(1,-\alpha)}\| \; \|g\mid \mathrm{Lip}_q^{(1,-\beta)}\| \\
&\qquad \times (|h|^2 \, |\log|h||^{\alpha+\beta} + |h| \, |\log|h||^{\alpha} + |h| \, |\log|h||^{\beta})
\end{aligned}$$

so that

$$\frac{\|\Delta_h(fg)\mid L_r\|}{|h|\,|\log|h||^{\alpha+\beta}} \leqslant C\,\|f\mid \mathrm{Lip}_p^{(1,-\alpha)}\|\,\|g\mid \mathrm{Lip}_q^{(1,-\beta)}\| \times(|h|+|\log|h||^{-\beta}+|\log|h||^{-\alpha}) \leqslant C'\,\|f\mid \mathrm{Lip}_p^{(1,-\alpha)}\|\,\|g\mid \mathrm{Lip}_q^{(1,-\beta)}\|$$

for all $h\in\mathbb{R}^n$, $0<|h|<1/2$. Likewise we have

$$\|fg\mid L_r\|\leqslant c\,\|f\mid L_p\|\,\|g\mid L_q\|\leqslant c'\,\|f\mid \mathrm{Lip}_p^{(1,-\alpha)}\|\,\|g\mid \mathrm{Lip}_q^{(1,-\beta)}\|,$$

so that by (4.1) we get (4.9). ∎

Remark 4.4. Note that one may strengthen the above argument in so far that

$$\mathrm{Lip}_p^{(1,-\alpha)}(\mathbb{R}^n)\cdot \mathrm{Lip}_q^{(1,-\beta)}(\mathbb{R}^n)\hookrightarrow \mathrm{Lip}_r^{(1,-\max(\alpha,\beta))}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}_r^{(1,-(\alpha+\beta))}(\mathbb{R}^n), \tag{4.10}$$

replacing (4.8). Then the constant c in the (modified) version of (4.9) depends upon the minimum of α and β, $c=c(\min(\alpha,\beta))$.

4.2. *Eigenvalue Estimates*

First we consider some model operator B of type $B=b\cdot b^{\Omega}(\cdot,D)$, acting in some (fractional) Sobolev space,

$$B\colon H_p^s(\Omega)\to H_p^s(\Omega), \tag{4.11}$$

where $b^{\Omega}(\cdot,D)$ belongs to some Hörmander class $S_{1,0}^{-\varkappa}$ (suitably adapted to the domain situation) and b is some multiplier function belonging to some (logarithmic) Lipschitz space. Recall that the Hörmander class $S_{1,\delta}^{\nu}$, $\nu\in\mathbb{R}$, $0\leqslant\delta\leqslant 1$, consists of all complex-valued C^∞ functions $(x,\xi)\mapsto a(x,\xi)$ on $\mathbb{R}^n\times\mathbb{R}^n$ such that for all multi-indices α, β there is a positive constant $c_{\alpha\beta}$ such that

$$|D_\xi^\alpha D_x^\beta a(x,\xi)|\leqslant c_{\alpha\beta}\langle\xi\rangle^{\nu-|\alpha|+\delta|\beta|}$$

for all $x,\xi\in\mathbb{R}^n$. It is well known that $a\in S_{1,0}^{-\varkappa}$ with $\varkappa\in\mathbb{R}$ maps $H_p^s(\mathbb{R}^n)$ into $H_p^{s+\varkappa}(\mathbb{R}^n)$ continuously, where $0<p<\infty$ and $s\in\mathbb{R}$, cf. [15, Theorem 5.2.3, p. 190]. Now let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. We may modify $a(\cdot,D)\colon H_p^s(\mathbb{R}^n)\to H_p^{s+\varkappa}(\mathbb{R}^n)$ by

$$re\circ a(\cdot,D)\circ ext\colon H_p^s(\Omega)\to H_p^{s+\varkappa}(\Omega),$$

where *re* is the natural restriction operator from $H_p^{s+\varkappa}(\mathbb{R}^n)$ to $H_p^{s+\varkappa}(\Omega)$ and *ext* a bounded linear extension from $H_p^s(\Omega)$ to $H_p^s(\mathbb{R}^n)$; see [33, 3.3.4, p. 201; 34,4.5, p. 225]. However, this result depends upon the way in which *ext* is constructed. To avoid any difficulties we may restrict ourselves to those spaces $H_p^s(\Omega)$ for which the characteristic function of Ω is a pointwise multiplier, that is, for which ext^{Ω}, given by

$$ext^{\Omega}\, f(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

is an extension map. Let $1 < p < \infty$; then the set of parameters $(\frac{1}{p}, s)$ such that ext^{Ω} is an extension map can be characterised in the following way,

$$1 < p < \infty, \qquad \frac{1}{p} - 1 < s < \frac{1}{p}; \tag{4.12}$$

we refer to [15, Proposition 5.2.3, p. 191; 33, 2.8.7, 3.3.2] for details. In other words, the operator

$$a^{\Omega}(\cdot, D) = re \circ a(\cdot, D) \circ ext^{\Omega}\colon H_p^s(\Omega) \to H_p^{s+\varkappa}(\Omega), \tag{4.13}$$

is bounded when $(\frac{1}{p}, s)$ satisfies assumption (4.12) and $a(\cdot, D) \in S_{1,0}^{-\varkappa}$, $\varkappa \in \mathbb{R}$. For simplicity, let us always assume $\Omega = U = \{x \in \mathbb{R}^n : |x| < 1\}$.

The crucial link between entropy numbers and eigenvalues of some compact operator is given by Carl's inequality. In particular, the setting is the following. Let A be a complex (quasi-) Banach space and $T \in \mathscr{L}(A)$ compact. Then the spectrum of T (apart from the point 0) consists only of eigenvalues of finite algebraic multiplicity. Let $\{\mu_k(T)\}_{k \in \mathbb{N}}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered such that

$$|\mu_1(T)| \geqslant |\mu_2(T)| \geqslant \cdots \geqslant 0.$$

Then Carl's inequality states that

$$\left(\prod_{m=1}^{k} |\mu_m(T)| \right)^{1/k} \leqslant \inf_{n \in \mathscr{N}} 2^{n/2k} e_n(T), \qquad k \in \mathbb{N}.$$

In particular, we have

$$|\mu_k(T)| \leqslant \sqrt{2}\, e_k(T). \tag{4.14}$$

This result was originally proved by Carl in [5] and Carl and Triebel in [8] when A is a Banach space. An extension to quasi-Banach spaces is proved in [15, Theorem 1.3.4].

Our first result deals with the case when $s = 1-(\varkappa - n/p) < 1$ in (4.11).

PROPOSITION 4.5. *Let* $1 < p < \infty$, $1 < \varkappa - \frac{n-1}{p} < 2$, *and* $\beta \geqslant 0$. *Assume* $b^U(\cdot, D) \in S^{-\varkappa}_{1,0}$, *and* $b \in \mathrm{Lip}_p^{(1, -\beta)}(U)$. *Let* $B = b \cdot b^U(\cdot, D)$. *Then*

$$B\colon H_p^{1-(\varkappa - n/p)}(U) \to H_p^{1-(\varkappa - n/p)}(U)$$

is compact. Denoting its eigenvalue sequence by $\{\mu_k(B)\}_{k \in \mathbb{N}}$, *there is some* $c > 0$ *such that for all* $k \in \mathbb{N}$,

$$|\mu_k(B)| \leqslant c\, \|b \mid \mathrm{Lip}_p^{(1, -\beta)}(U)\|\, k^{-\varkappa/n} (\log\langle k\rangle)^{2+1/p+(1/p-1/2)_+}. \tag{4.15}$$

Furthermore, we gain some additional smoothness when applying B, for

$$Im(B) \subset \mathrm{Lip}_p^{(1, -\max(\beta, 1/p'))}(U) \hookrightarrow H_p^{1-(\varkappa - n/p)}(U). \tag{4.16}$$

Proof. We decompose the above operator B into $B = id_2 \circ b \circ id_1 \circ b^U(\cdot, D)$, where

$$\begin{aligned} b^U(\cdot, D)\colon H^{1-(\varkappa - n/p)}(U) &\to H^{1+n/p}(U) \\ id_1\colon H_p^{1+n/p}(U) &\to \mathrm{Lip}^{(1,-\alpha)}(U) \\ b\colon \mathrm{Lip}^{(1,-\alpha)}(U) &\to \mathrm{Lip}_p^{(1,-\alpha)}(U) \\ id_2\colon \mathrm{Lip}_p^{(1,-\alpha)}(U) &\to H_p^{1-(\varkappa - n/p)}(U), \end{aligned} \tag{4.17}$$

and $\alpha > 0$ might be chosen sufficiently large, say $\alpha > \max(2 + \frac{1}{p} + (\frac{1}{p} - \frac{1}{2})_+, \beta)$. Note that by our assumption about $\varkappa$ we get that $(\frac{1}{p}, s) = (\frac{1}{p}, 1 - (\varkappa - \frac{n}{p}))$ satisfies assumption (4.12). Thus the first embedding is established by (4.13), whereas we get by the elementary embedding $H_p^{1+n/p}(U) \hookrightarrow B^{1+n/p}_{p, \max(p,2)}(U)$ and Theorem 3.11 for id_1 that

$$\begin{aligned} e_k(id_1) &\leqslant c e_k(id\colon B^{1+n/p}_{p,\max(p,2)}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)) \\ &\leqslant c' k^{-1/p} (\log\langle k\rangle)^{-\alpha + \max(1/p', 1/2) + 1 + 2/p} \\ &= c' k^{-1/p} (\log\langle k\rangle)^{-\alpha + 2 + 1/p + (1/p - 1/2)_+}. \end{aligned} \tag{4.18}$$

Our assumption about the multiplier function b and Proposition 4.3, i.e., (4.10), yield the continuity of the third map in (4.17). Concerning id_2 we have by Proposition 4.2 (i) and [15, Theorem 3.3.2, p. 105] that

$$\begin{aligned} e_k(id_2) &\leqslant c\lambda^{-\alpha} e_k(id\colon B^{1-\lambda}_{p,\infty}(U) \to H_p^{1-(\varkappa - n/p)}(U)) \\ &\leqslant c' \lambda^{-\alpha} k^{\lambda/n} k^{-\varkappa/n + 1/p} \\ &\leqslant C (\log\langle k\rangle)^{\alpha}\, k^{-\varkappa/n + 1/p}, \end{aligned} \tag{4.19}$$

using again the minimising argument (3.41). Thus (4.18) and (4.19), together with Carl's inequality (4.14) prove (4.15). We deal with (4.16). Note that id_1 in (4.17) might be replaced by $id'_1: H^{1+n/p}_p(U) \to \mathrm{Lip}^{(1,-1/p')}_p(U)$, recall $H^s_p = F^s_{p,2}$, $s \in \mathbb{R}$, $1<p<\infty$, and Theorem 2.1(i). (This is also the original Brézis–Wainger result, see Remark 2.3.) Hence b: $\mathrm{Lip}^{(1,-1/p')}_p(U) \to \mathrm{Lip}^{(1,-\max(\beta,1/p'))}_p(U)$ which completes the proof. ▮

Assume now $1-(\varkappa-n/p)<s<1$ in (4.11). Then we get a sharper result than the corresponding one in Proposition 4.5.

PROPOSITION 4.6. *Let* $1<p<\infty$, $\varkappa>1+\frac{n-1}{p}$, $\max(1-(\varkappa-\frac{n}{p}), \frac{1}{p}-1)<s<\frac{1}{p}$, *and* $\beta \geqslant 0$. *Assume* $b^U(\cdot, D) \in S^{-\varkappa}_{1,0}$, *and* $b \in \mathrm{Lip}^{(1,-\beta)}_p(U)$. *Then* $B = b \cdot b^U(\cdot, D)$ *acts compactly from* $H^s_p(U)$ *into itself, with*

$$|\mu_k(B)| \leqslant c\, \|b \mid \mathrm{Lip}^{(1,-\beta)}_p(U)\|\, k^{-\varkappa/n} \tag{4.20}$$

for some $c>0$ *and all* $k \in \mathbb{N}$.

Proof. We modify the decomposition (4.17) in the following way:

$$b^U(\cdot, D)\colon H^s_p(U) \to H^{s+\varkappa}_p(U)$$
$$id_1\colon H^{s+\varkappa}_p(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)$$
$$b\colon \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathrm{Lip}^{(1,-\alpha)}_p(U)$$
$$id_2\colon \mathrm{Lip}^{(1,-\alpha)}_p(U) \to H^s_p(U).$$

Our assumption about s implies that $b^U(\cdot, D)$ is bounded by (4.12), (4.13). Note that

$$e_k(id_1) \leqslant c k^{-(s+\varkappa-1)/n} (\log\langle k\rangle)^{-\alpha}$$

by (3.55) and $s+\varkappa>1+n/p$, where $\alpha>0$ is at our disposal. We may choose $\alpha>\max(\beta, 1/p')$. Furthermore, (1.6) and [15, Theorem 3.3.3/2, p. 118] imply

$$\begin{aligned} e_k(id_2) &\leqslant c\lambda^{-\alpha} e_k(id\colon B^{1-\lambda}_{p,\infty}(U) \to H^s_p(U)) \leqslant c'\lambda^{-\alpha} k^{\lambda/n} k^{-(1-s)/n} \\ &\leqslant C k^{-(1-s)/n} (\log\langle k\rangle)^{\alpha}, \end{aligned} \tag{4.21}$$

where the last estimate is covered by the minimising argument (3.41) again. Thus we obtain by Carl's inequality (4.14) and the multiplicativity of entropy numbers that

$$|\mu_k(B)| \leqslant \sqrt{2}\, e_k(B) \leqslant c\, \|b \mid \mathrm{Lip}^{(1,-\beta)}_p(U)\|\, k^{-\varkappa/n}.$$

Note that one has some smoothness result parallel to (4.16), too, that is,

$$Im(B) \subset \mathrm{Lip}_p^{(1,\, -\max(\beta,\, 1/p'))}(U) \hookrightarrow H_p^s(U). \quad \blacksquare$$

We end this section with some application for an elliptic differential operator. We closely follow [15, Sects. 5.2.2, 5.2.4, pp. 187, 192]. Recall that U is the unit ball in $\mathbb{R}^n$. Let A be a properly elliptic operator,

$$Af = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, D^\alpha f, \qquad \text{where each} \quad a_\alpha \in C^\infty(\bar{U}), \tag{4.22}$$

and suppose there are boundary operators

$$B_j f = \sum_{|\alpha| \leqslant \ell_j} b_{j,\alpha}(x)\, D^\alpha f, \qquad \text{where each} \quad b_{j,\alpha} \in C^\infty(\partial U), \tag{4.23}$$

with $j = 1, ..., m$, $m \in \mathbb{N}$, and $0 \leqslant \ell_1 < \cdots < \ell_m \leqslant 2m-1$, which form a normal system satisfying the complementing condition. Then $\{A; B_1, ..., B_m\}$ is called a *regular elliptic system*; see [33, Sect. 4.1.2, p. 213] for details. We assume that the problem

$$\begin{aligned} Af &= 0 \qquad \text{in } U, \\ B_j f &= 0 \qquad \text{on } \partial U, \qquad j = 1, ..., m, \end{aligned} \tag{4.24}$$

has only the trivial C^∞ solution, cf. [33, p. 231]. We make use of the following basic result: *Suppose that* $1 < p < \infty$, $s \geqslant 0$. *Then* A *maps*

$$\{f \in H_p^{s+2m}(U) : B_j f = 0 \text{ on } \partial U \text{ for } j = 1, ..., m\}$$

isomorphically onto $H_p^s(U)$.

We refer to [1; 16; 33, Chap. 4, p. 233] for proofs, the latter two dealing with more general settings. The above version can be found (as a special case) in [15, Sect. 5.2.2, p. 187]. Now (4.24) implies that 0 does not belong to the spectrum $\sigma(A)$ of A, and consequently $\sigma(A)$ consists of isolated eigenvalues of finite algebraic multiplicity. We study the map

$$Bf = b_2 A^{-1} b_1 f, \tag{4.25}$$

where A^{-1} is the inverse of the above operator A, $b_1 \in L_r(U)$ and $b_2 \in \mathrm{Lip}_p^{(1,\, -\beta)}(U)$.

Corollary 4.7. *Let* $1 < p < \infty$, $p' < r \leqslant \infty$, *and* $m \in \mathbb{N}$ *with* $2m > 1 + n(\frac{1}{p} + \frac{1}{r})$. *Let* A^{-1} *be the inverse of the elliptic operator described above.*

Assume that $b_1 \in L_r(U)$ *and* $b_2 \in \mathrm{Lip}_p^{(1,-\beta)}(U)$ *for some* $\beta \geq 0$. *Then* B, *given by* (4.25), *acts compactly in* $L_p(U)$, *and*

$$|\mu_k(B)| \leq c \, \|b_1 \mid L_r(U)\| \, \|b_2 \mid \mathrm{Lip}_p^{(1,-\beta)}(U)\| \, k^{-2m/n}$$

for some $c > 0$ *and all* $k \in \mathbb{N}$.

Proof. We use the decomposition $B = id_2 \circ b_2 \circ id_1 \circ A^{-1} \circ b_1$ with

$$\begin{aligned} b_1 &: L_p(U) \to L_q(U), \qquad \text{where} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \\ A^{-1} &: L_q(U) \to H_q^{2m}(U) \\ id_1 &: H_q^{2m}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U) \\ b_2 &: \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathrm{Lip}_p^{(1,-\alpha)}(U) \\ id_2 &: \mathrm{Lip}^{(1,-\alpha)}(U) \to L_p(U). \end{aligned}$$

The first embedding is a consequence of Hölder's inequality for $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, the rest is similar to the proof of Proposition 4.6. Note that our assumptions imply that $1 < q < \infty$ and that id_1 is compact. Hence the upper estimate for $e_k(id_1)$ is covered by (3.55), whereas the corresponding one for $e_k(id_2)$ is given by (4.21) (with $s = 0$). The assertion follows by the multiplicativity of entropy numbers and Carl's inequality (4.14). ▮

Note that one can derive some smoothness assertion parallel to (4.16) in the situation described above, too. In particular, our above assumptions imply

$$Im(B) \subset \mathrm{Lip}_p^{(1,-\max(\beta,1/p'-1/r))}(U) \hookrightarrow L_p(U).$$

This result seems also new.

Remark 4.8. Likewise one can study the operator B, given by $B = b_2 A^{-\varkappa} b_1$, $0 < \varkappa \leq 1$, where the (inverse of the) above operator A is now being replaced by some fractional power $A^{\varkappa}$ of a regular elliptic operator A. We do not wan to discuss this setting here, but details may be found in [15, Sect. 5.2.4, p. 198]. Modifications in order to obtain results of the above type are obvious.

We have described two model operators (4.11) and (4.25) in a little detail now. However, our main intention (in this section) was to indicate how to use our results about entropy numbers when dealing with eigenvalue estimates. Further interesting applications are clearly possible.

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